

# CONDITIONAL ERGODICITY IN INFINITE DIMENSION\*

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The goal of this paper is to develop a general method to establish conditional ergodicity of infinite-dimensional Markov chains. Given a Markov chain in a product space, we aim to understand the ergodic properties of its conditional distributions given one of the components. Such questions play a fundamental role in the ergodic theory of nonlinear filters. In the setting of Harris chains, conditional ergodicity has been established under general nondegeneracy assumptions. Unfortunately, Markov chains in infinite-dimensional state spaces are rarely amenable to the classical theory of Harris chains due to the singularity of their transition probabilities, while topological and functional methods that have been developed in the ergodic theory of infinite-dimensional Markov chains are not well suited to the investigation of conditional distributions. We must therefore develop new measure-theoretic tools in the ergodic theory of Markov chains that enable the investigation of conditional ergodicity for infinite dimensional or weak-\* ergodic processes. To this end, we first develop local counterparts of zero-two laws that arise in the theory of Harris chains. These results give rise to ergodic theorems for Markov chains that admit asymptotic couplings or that are locally mixing in the sense of H. Föllmer, and to a non-Markovian ergodic theorem for stationary absolutely regular sequences. We proceed to show that local ergodicity is inherited by conditioning on a nondegenerate observation process. This is used to prove stability and unique ergodicity of the nonlinear filter. Finally, we show that our abstract results can be applied to infinite-dimensional Markov processes that arise in several settings, including dissipative stochastic partial differential equations, stochastic spin systems, and stochastic differential delay equations.

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**1. Introduction.** The classical ergodic theory of Markov chains in general state spaces has achieved a rather definitive form in the theory of Harris chains [34, 30, 29], which provides necessary and sufficient conditions for the convergence of the transition probabilities in total variation to an invariant measure. While this theory is formulated in principle for any measurable state space, it is well known that its applicability extends in practice mainly to finite-dimensional situations. In infinite dimension, the transition probabilities from different initial conditions tend to be mutually singular even in the most trivial examples, so that total variation convergence is out of the question. For this reason, many infinite-dimensional Markov processes, including stochastic partial differential equations, interacting particle systems, and stochastic equations with memory, lie outside the scope of the classical theory. Instead, a variety of different approaches, including topological [9, 17, 19], functional [26, 18], coupling and duality [24] methods, have been employed to investigate the ergodicity of infinite-dimensional models.

The goal of this paper is to investigate questions of *conditional* ergodicity in infinite dimension. Consider a Markov chain  $(X_n, Y_n)_{n \geq 0}$  taking values in a product space  $E \times F$  (continuous time processes are considered analogously). The aim of conditional ergodic theory is to understand the ergodic properties of one component of the process  $(X_n)_{n \geq 0}$  under the conditional distribution given the other component  $(Y_n)_{n \geq 0}$ . Even when the process  $(X_n, Y_n)_{n \geq 0}$  is ergodic, the inheritance of ergodicity under conditioning is

far from obvious and does not always hold. The history of such problems dates back to an erroneous result of H. Kunita [21], where the inheritance of ergodicity was taken for granted (see [43] and the references therein). The long-standing problem of establishing conditional ergodicity under general assumptions was largely resolved in [40, 39], where it is shown that the inheritance of ergodicity holds under a mild nondegeneracy assumption when  $(X_n, Y_n)_{n \geq 0}$  is a Harris chain. Numerous other results in this area, both of a qualitative and quantitative nature, are reviewed in [8]. All these results are however essentially restricted to the setting of Harris chains, so that their applicability to infinite-dimensional models is severely limited. In this paper, we develop the first results of this kind that are generally applicable beyond the Harris setting and, in particular, that allow to establish conditional ergodicity in a wide range infinite-dimensional models.

To give a flavor of the type of problems that our theory will address, let us briefly describe one example that will be given in section 5 below. Consider the velocity field  $u$  of a fluid that is modeled as a Navier-Stokes equation

$$du = \{\nu \Delta u - (u \cdot \nabla)u - \nabla p\} dt + d\tilde{w}, \quad \nabla \cdot u = 0$$

with white in time, spatially smooth random forcing  $d\tilde{w}$ . At regular time intervals  $t_n = n\delta$  the velocity field is sampled at the spatial locations  $z_1, \dots, z_r$  with some additive Gaussian noise  $\xi_n$ , which yields the observations

$$Y_n^i = u(t_n, z_i) + \xi_n^i, \quad i = 1, \dots, r.$$

Such models arise naturally in data assimilation problems [38]. The process  $(X_n, Y_n)_{n \geq 0}$  with  $X_n = u(t_n, \cdot)$  is an infinite-dimensional Markov chain. Classical ergodicity questions include the existence and uniqueness of an invariant probability  $\lambda$ , and the convergence to equilibrium property

$$|\mathbf{E}^x[f(X_n)] - \lambda(f)| \xrightarrow{n \rightarrow \infty} 0$$

for a sufficiently large class of functions  $f$  and initial conditions  $x$ . Such questions are far from straightforward for Navier-Stokes equations and have formed a very active area of research in recent years (see, for example, [16, 28, 20]). In contrast, we are interested in the question of conditional ergodicity

$$|\mathbf{E}^x[f(X_n) | \mathcal{F}_{0,\infty}^Y] - \mathbf{E}^\lambda[f(X_n) | \mathcal{F}_{0,\infty}^Y]| \xrightarrow{n \rightarrow \infty} 0$$

(where  $\mathcal{F}_{m,n}^Y = \sigma\{Y_m, \dots, Y_n\}$ ), or, more importantly, its causal counterpart

$$|\mathbf{E}^x[f(X_n) | \mathcal{F}_{0,n}^Y] - \mathbf{E}^\lambda[f(X_n) | \mathcal{F}_{0,n}^Y]| \xrightarrow{n \rightarrow \infty} 0$$

which corresponds to stability of the nonlinear filter  $\pi_n^\mu = \mathbf{P}^\mu[X_n \in \cdot | \mathcal{F}_{0,n}^Y]$ . In contrast to convergence to equilibrium of the underlying model, conditional ergodicity properties yield convergence to equilibrium of the estimation error of the model given the observations [21] or the long-term stability of the conditional distributions to perturbations (such as those that arise in the investigation of numerical filtering algorithms), cf. [42]. The interplay between ergodicity and conditioning is of intrinsic interest in probability theory and in measurable dynamics, where it is closely related to notions of relative mixing [35], and lies at the heart of stability problems that arise in data assimilation and nonlinear filtering. The main results of this paper will allow us to establish conditional ergodicity in a wide range of infinite-dimensional models, including dissipative stochastic partial differential equations such as the above Navier-Stokes model, stochastic spin systems, and stochastic differential delay equations (detailed examples are given in section 5).

One of the main difficulties in the investigation of conditional ergodicity is that conditioning on an infinite observation sequence  $\mathcal{F}_{0,\infty}^Y$  is a very singular operation. Under the conditional distribution, the unobserved process  $(X_n)_{n \geq 0}$  remains a Markov chain, albeit an inhomogeneous one with random transition probabilities depending on the realized path of the observations  $(Y_n)_{n \geq 0}$  (in the stationary case this is a Markov chain in a random environment in the sense of Cogburn and Orey [7, 31]). These conditional transition probabilities are defined abstractly as regular conditional probabilities, but no explicit equations are available even in the simplest examples. There is therefore little hope of analyzing the properties of the conditional chain “by hand,” and one must find a way to deduce the requisite ergodic properties from their unconditional counterparts. On the other hand, conditioning is an essentially measure-theoretic operation, and it is unlikely that the most fruitful approaches to ergodic theory in infinite dimension, such as topological properties or functional inequalities, are preserved by the conditional distributions. To move beyond the setting of Harris chains, we therefore aim to find a way to encode such weak ergodic properties in a measure-theoretic fashion that can be shown to be preserved under conditioning.

A central insight of this paper is that certain basic elements of the classical theory admit *local* formulations that do not rely on the Markov property. The simplest of these is a local zero-two law (section 2.1) that characterizes, for a given  $E$ -valued Markov chain  $(X_n)_{n \geq 0}$  and measurable map  $\iota : E \rightarrow E'$  to another space  $E'$ , the following total variation ergodic property:

$$\|\mathbf{P}^x[(\iota(X_k))_{k \geq n} \in \cdot] - \mathbf{P}^{x'}[(\iota(X_k))_{k \geq n} \in \cdot]\| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } x, x' \in E.$$

If  $\iota$  is injective, then this reduces to the ergodic property of a Harris chain.

By choosing different functions  $\iota$ , however, we will find that such results are applicable far beyond the setting of Harris chains. Let us emphasize that when  $\iota$  is not injective the process  $(\iota(X_n))_{n \geq 0}$  is generally not Markov, so that our local ergodic theorems are fundamentally non-Markovian in nature.

In certain cases, this local notion of ergodicity can be applied directly to infinite-dimensional Markov chains. When the entire chain does not converge to equilibrium in total variation, it may still be the case that each finite-dimensional projection of the chain converges in the above sense. To our knowledge, this *local mixing* property was first proposed by H. Föllmer [15] in the context of interacting particle systems; a similar idea appears in [27] for stochastic Navier-Stokes equations with sufficiently nondegenerate forcing. By choosing  $\iota$  to be a finite-dimensional projection, we obtain a very useful characterization of the local mixing property (section 2.2). Our results can also be applied directly to non-Markovian processes: for example, we will obtain a non-Markovian ergodic theorem that provides an apparently new characterization of stationary absolutely regular sequences (section 2.3).

While local mixing can be verified in various infinite-dimensional models, this generally requires a fair amount of nondegeneracy. In truly degenerate situations, we introduce another idea that exploits topological properties of the model (section 2.4). In dissipative models and in many other Markov chains that converge weakly to equilibrium, it is possible to construct a coupling of two copies  $X_n, X'_n$  of the chain such that  $d(X_n, X'_n) \rightarrow 0$  (cf. [17]). Of course, this need not imply any form of total variation convergence. Consider, however, the perturbed process  $f(X_n) + \eta_n$  where  $f : E \rightarrow \mathbb{R}$  is a Lipschitz function and  $(\eta_n)_{n \geq 0}$  is an i.i.d. sequence of auxiliary Gaussian random variables. When the asymptotic coupling converges sufficiently rapidly, the process  $(f(X_n) + \eta_n)_{n \geq 0}$  will be ergodic in the above total variation sense by the Kakutani theorem. We have thus transformed a topological property into a measure-theoretic one, which is amenable to our local ergodic theorems by considering the augmented Markov chain  $(X_n, \eta_n)_{n \geq 0}$  with  $\iota(x, \eta) = f(x) + \eta$ . The added noise can ultimately be deconvolved, which yields weak-\* ergodic theorems for the original chain  $(X_n)_{n \geq 0}$  by purely measure-theoretic means.

The local ergodic theorems developed in section 2 are of independent interest. However, the full benefit of our approach emerges in the development of the conditional ergodic theory that is undertaken in sections 3 and 4. First, we develop in section 3.1 a conditional counterpart to the local zero-two law that characterizes the conditional absolute regularity property of a stationary (non-Markovian) sequence. The remainder of section 3 is devoted to the inheritance problem. In short, we show that under a generalization of the nondegeneracy assumption on the observations that was introduced

in [40, 39], the local ergodicity property is inherited when we condition on the observed component of the model. Together with the ideas developed in section 2, this allows us to obtain various filter stability results in section 4. After introducing the relevant setting and notations in section 4.1, we first develop a general local filter stability theorem in section 4.2. In section 4.3, we give concrete filter stability theorems for Markov chains that are locally mixing or that admit asymptotic couplings. We also investigate unique ergodicity of the filtering process in the spirit of [21]. Finally, in section 4.4, we extend our main results to Markov processes in continuous time. Our general approach in these sections is inspired by the ideas developed in [40] in the Harris setting. However, as is explained in section 3, the approach used in [40, 39] relies crucially on the Markov property, and the same method can therefore not be used in the local setting. Instead, we develop here a new (and in fact somewhat more direct) method for establishing the inheritance property that does not rely on Markov-specific arguments.

To illustrate the wide applicability of our results, we develop in section 5 several infinite-dimensional examples that were already mentioned above. Our aim is to demonstrate that the assumptions of our main results can be verified in several quite distinct settings. In order not to unduly lengthen the paper, we have restricted attention to a number of examples whose ergodic properties are readily verified using existing results in the literature.

Let us conclude the introduction by briefly highlighting two directions that are not addressed in this paper. First, we emphasize that all the results in this paper, which rely at their core on martingale convergence arguments, are qualitative in nature. The development of quantitative filter stability results is an interesting problem, and this remains challenging even in finite-dimensional models (cf. [8] and the references therein). Second, let us note that the crucial assumption that ensures inheritance of ergodicity under conditioning is nondegeneracy of the observations (Assumption 4.3). This assumption states, in essence, that the conditional distribution of the observations on a finite time interval, given the unobserved process, has a transition density. In practice, this implies that while the unobserved process  $X_n$  may be infinite-dimensional, the observations  $Y_n$  must typically be finite-dimensional in order to apply our results. While the present setting covers a wide range of models of practical interest, conditional ergodicity problems with infinite-dimensional observations are of significant interest in their own right and require separate consideration. In the latter setting new probabilistic phenomena can arise; such issues will be discussed elsewhere.

**2. Local ergodic theorems.** The goal of this section is to develop a number of simple but powerful measure-theoretic ergodic theorems that are applicable beyond the classical setting of Harris chains [34]. Our main tools are the *local* zero-two laws developed in section 2.1. In the following subsections, it is shown how these results can be applied in various different settings. In section 2.2, we consider a notion of local mixing for Markov chains, due to H. Föllmer [15], that provides a natural measure-theoretic generalization of Harris chains to the infinite-dimensional setting. In section 2.3, we obtain an ergodic theorem for non-Markov processes that yields a new characterization of stationary absolutely regular sequences. Finally, in section 2.4, we show how these results can be combined with the notion of asymptotic coupling (see, for example, [17]) to obtain ergodic theorems in the weak convergence topology by purely measure-theoretic means.

Throughout this section, we will work in the following canonical setup. Let  $(E, \mathcal{E})$  be a measurable space, and let  $(X_k)_{k \in \mathbb{Z}}$  be the  $E$ -valued coordinate process defined on the canonical path space  $(\Omega, \mathcal{F})$ . That is, we define  $\Omega = E^{\mathbb{Z}}$ ,  $\mathcal{F} = \mathcal{E}^{\mathbb{Z}}$ , and  $X_k(\omega) = \omega(k)$ . We define for  $m < n$

$$X_{m,n} = (X_k)_{m \leq k \leq n}, \quad \mathcal{F}_{m,n} = \sigma\{X_{m,n}\}, \quad \mathcal{F}_+ = \mathcal{F}_{0,\infty}, \quad \mathcal{F}_- = \mathcal{F}_{-\infty,0}.$$

We also define the canonical shift  $\Theta : \Omega \rightarrow \Omega$  as  $\Theta(\omega)(n) = \omega(n+1)$ .

We will denote by  $\mathcal{P}(Z)$  the set of probability measures on a measurable space  $(Z, \mathcal{Z})$ , and for  $\mu, \nu \in \mathcal{P}(Z)$  we denote by  $\|\mu - \nu\|_{\mathcal{Z}_0}$  the total variation of the signed measure  $\mu - \nu$  on the  $\sigma$ -field  $\mathcal{Z}_0 \subseteq \mathcal{Z}$ , that is,

$$\|\mu - \nu\|_{\mathcal{Z}_0} = 2 \sup_{A \in \mathcal{Z}_0} |\mu(A) - \nu(A)|.$$

For simplicity, we will write  $\|\mu - \nu\| = \|\mu - \nu\|_{\mathcal{Z}}$ . Let us recall that if  $K, K'$  are finite kernels and if  $\mathcal{Z}$  is countably generated, then  $x \mapsto \|K(x, \cdot) - K'(x, \cdot)\|$  is measurable (see, for example, [40, Lemma 2.4]). In this setting, we have  $\|\int \{K(x, \cdot) - K'(x, \cdot)\} \mu(dx)\| \leq \int \|K(x, \cdot) - K'(x, \cdot)\| \mu(dx)$  by Jensen's inequality. These facts will be used repeatedly throughout the paper.

**2.1. Local zero-two laws.** Let  $P : E \times \mathcal{E} \rightarrow [0, 1]$  be a transition kernel on  $(E, \mathcal{E})$ , and denote by  $\mathbf{P}^\mu$  be the probability measure on  $\mathcal{F}_+$  such that  $(X_k)_{k \geq 0}$  is Markov with transition kernel  $P$  and initial law  $X_0 \sim \mu \in \mathcal{P}(E)$ . If  $P$  is Harris and aperiodic, the Markov chain is ergodic in the sense that

$$\|\mu P^n - \nu P^n\| = \|\mathbf{P}^\mu - \mathbf{P}^\nu\|_{\mathcal{F}_{n,\infty}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } \mu, \nu \in \mathcal{P}(E)$$

(cf. [34, Theorem 6.2.2]). Unfortunately, this mode of convergence can be restrictive in complex models. For example, when the state space  $E$  is infinite-dimensional, such strong convergence will rarely hold: it is often the case in this setting that  $\mu P^n \perp \nu P^n$  for all  $n \geq 0$  (cf. section 2.2).

At the heart of this paper lies a simple idea. When total variation convergence of the full chain fails, it may still be the case that total variation convergence holds when the chain restricted to a smaller  $\sigma$ -field  $\mathcal{E}^0 \subset \mathcal{E}$ : that is, we intend to establish convergence of  $\iota(X_k)$  where  $\iota : (E, \mathcal{E}) \rightarrow (E, \mathcal{E}^0)$  is the identity map. As will become clear in the sequel, such *local* total variation convergence is frequently sufficient to deduce convergence of the full chain in a weaker probability distance, while at the same time admitting a powerful measure-theoretic ergodic theory that will be crucial for the study of conditional ergodicity in complex models.

The key results of this section are a pair of local zero-two laws that characterize the local total variation convergence of Markov processes. Let us fix  $\mathcal{E}^0 \subseteq \mathcal{E}$  throughout this section, and define the  $\sigma$ -fields

$$\mathcal{F}_{m,n}^0 = \bigvee_{m \leq k \leq n} X_k^{-1}(\mathcal{E}^0), \quad m < n.$$

A central role will be played by the local tail  $\sigma$ -field

$$\mathcal{A}^0 = \bigcap_{n \geq 0} \mathcal{F}_{n,\infty}^0.$$

Finally, for  $x \in E$  we will denote for simplicity  $\mathbf{P}^x = \mathbf{P}^{\delta_x}$ .

It is important to note that the local process  $\iota(X_k)$  is generally not Markov, so that the marginal distribution at a fixed time does not determine the future of this process. Thus one cannot restrict attention to the marginal distance  $\|\mu P^n - \nu P^n\|_{\mathcal{E}^0}$ , but one must instead consider the entire infinite future  $\|\mathbf{P}^\mu - \mathbf{P}^\nu\|_{\mathcal{F}_{n,\infty}^0}$ . Of course, when  $\mathcal{E}^0 = \mathcal{E}$ , these notions coincide.

**THEOREM 2.1** (Local zero-two law). *The following are equivalent.*

1. *The Markov chain is locally ergodic:*

$$\|\mathbf{P}^\mu - \mathbf{P}^\nu\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for every } \mu, \nu \in \mathcal{P}(E).$$

2. *The local tail  $\sigma$ -field is trivial:*

$$\mathbf{P}^\mu(A) \in \{0, 1\} \quad \text{for every } A \in \mathcal{A}^0 \text{ and } \mu \in \mathcal{P}(E).$$

3. *The Markov chain is locally irreducible: there exists  $\alpha > 0$  such that*

$$\forall x, x' \in E, \quad \exists n \geq 0 \quad \text{such that} \quad \|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{F}_{n,\infty}^0} \leq 2 - \alpha.$$

Zero-two laws of this type appear naturally in the theory of Harris chains [34, 11, 32]. It is somewhat surprising that the Markov property proves to be inessential in the proof, which enables the present local formulation.



PROOF OF THEOREM 2.1. We prove  $2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 2$ .

( $2 \Rightarrow 1$ ). Assumption 2 implies that  $\mathbf{P}^\mu(A) = \mathbf{P}^\nu(A)$  for all  $A \in \mathcal{A}^0$  (if not, then  $\mathbf{P}^\rho(A) = 1/2$  for  $\rho = (\mu + \nu)/2$ , a contradiction). Therefore

$$\|\mathbf{P}^\mu - \mathbf{P}^\nu\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} \|\mathbf{P}^\mu - \mathbf{P}^\nu\|_{\mathcal{A}^0} = 0.$$

( $1 \Rightarrow 3$ ). This is obvious.

( $3 \Rightarrow 2$ ). Assume that condition 2 does not hold. Then there exists  $A \in \mathcal{A}^0$  and  $\mu \in \mathcal{P}(E)$  such that  $0 < \mathbf{P}^\mu(A) < 1$ . Define  $f = \mathbf{1}_A - \mathbf{1}_{A^c}$ , and note that

$$\mathbf{E}^{X_n}[f \circ \Theta^{-n}] = \mathbf{E}^\mu[f | \mathcal{F}_{0,n}] \xrightarrow{n \rightarrow \infty} f \quad \mathbf{P}^\mu\text{-a.s.}$$

Define the probability measure  $\mathbf{Q}$  on  $\Omega \times \Omega$  as  $\mathbf{P}^\mu \otimes \mathbf{P}^\mu$ , and denote by  $(X_n, X'_n)_{n \geq 0}$  the coordinate process on  $\Omega \times \Omega$ . Fix  $\alpha > 0$ . Then

$$\mathbf{Q}[|\mathbf{E}^{X_n}[f \circ \Theta^{-n}] - \mathbf{E}^{X'_n}[f \circ \Theta^{-n}]| > 2 - \alpha] \xrightarrow{n \rightarrow \infty} 2 \mathbf{P}^\mu(A) \mathbf{P}^\mu(A^c) > 0.$$

Thus there exist  $N \geq 0$  and  $x, x' \in E$  such that  $|\mathbf{E}^x[f \circ \Theta^{-N}] - \mathbf{E}^{x'}[f \circ \Theta^{-N}]| > 2 - \alpha$ . But note that  $|f| \leq 1$  and  $f \circ \Theta^{-N}$  is  $\mathcal{A}^0$ -measurable. Therefore,

$$\|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{F}_{n,\infty}^0} \geq \|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{A}^0} \geq |\mathbf{E}^x[f \circ \Theta^{-N}] - \mathbf{E}^{x'}[f \circ \Theta^{-N}]| > 2 - \alpha$$

for all  $n \geq 0$ . As  $\alpha > 0$  is arbitrary, condition 3 is contradicted.  $\square$

The characterization in Theorem 2.1 does not require the existence of an invariant probability. However, when such a probability exists, we can obtain a useful stationary variant of the local zero-two law that will be proved next. The advantage of the stationary zero-two law is that it does not require uniform control in condition 3. On the other hand, the resulting convergence only holds for almost every initial condition.

THEOREM 2.2 (Local stationary zero-two law). *Suppose  $\mathcal{E}^0$  is countably generated. Given a  $P$ -invariant probability  $\lambda$ , the following are equivalent.*

1. *The Markov chain is a.e. locally ergodic:*

$$\|\mathbf{P}^x - \mathbf{P}^\lambda\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \lambda\text{-a.e. } x,$$

or, equivalently,

$$\|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \lambda \otimes \lambda\text{-a.e. } (x, x').$$

2. *The local tail  $\sigma$ -field is a.e. trivial:*

$$\mathbf{P}^x(A) = \mathbf{P}^x(A)^2 = \mathbf{P}^{x'}(A) \quad \forall A \in \mathcal{A}^0, \quad \lambda \otimes \lambda\text{-a.e. } (x, x').$$

3. *The Markov chain is a.e. locally irreducible:*

for  $\lambda \otimes \lambda$ -a.e.  $(x, x')$ ,  $\exists n \geq 0$  such that  $\|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{G}_{n,\infty}^0} < 2$ .

PROOF. The equivalence of the two statements of condition 1 follows from

$$\begin{aligned} \|\mathbf{P}^x - \mathbf{P}^\lambda\|_{\mathcal{G}_{n,\infty}^0} &\leq \int \|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{G}_{n,\infty}^0} \lambda(dx'), \\ \|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{G}_{n,\infty}^0} &\leq \|\mathbf{P}^x - \mathbf{P}^\lambda\|_{\mathcal{G}_{n,\infty}^0} + \|\mathbf{P}^{x'} - \mathbf{P}^\lambda\|_{\mathcal{G}_{n,\infty}^0}. \end{aligned}$$

The proofs of  $2 \Rightarrow 1 \Rightarrow 3$  are identical to the corresponding proofs in Theorem 2.1. It therefore remains to prove  $3 \Rightarrow 2$ . To this end, define

$$\beta_n(x, x') = \|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{G}_{n,\infty}^0}, \quad \beta(x, x') = \lim_{n \rightarrow \infty} \beta_n(x, x').$$

The limit in the definition of  $\beta$  exists pointwise as  $\beta_n$  is decreasing in  $n$ . Moreover, as  $\mathcal{E}^0$  is countably generated, the maps  $\beta_n$  and  $\beta$  are measurable.

Define  $\mathbf{Q}^{x,x'} = \mathbf{P}^x \otimes \mathbf{P}^{x'}$  on  $\Omega \times \Omega$ . Note that  $\beta_n \leq (P \otimes P)\beta_{n-1}$  pointwise by Jensen's inequality, so  $\beta \leq (P \otimes P)\beta$  by dominated convergence. In particular, as  $\lambda \otimes \lambda$  is  $P \otimes P$ -invariant, this implies  $\beta = (P \otimes P)\beta$   $\lambda \otimes \lambda$ -a.e. Thus there is a measurable set  $H' \subseteq \Omega \times \Omega$  with  $(\lambda \otimes \lambda)(H') = 1$  such that

$$\mathbf{Q}^{x,x'}[\beta(x, x') = \beta(X_n, X'_n) \text{ for all } n \geq 0] = 1 \quad \text{for all } (x, x') \in H'.$$

In the remainder of the proof, we assume that condition 3 holds, and we fix a measurable set  $H \subseteq H'$  with  $(\lambda \otimes \lambda)(H) = 1$  such that

$$\forall (x, x') \in H, \quad \exists n \geq 0 \quad \text{such that} \quad \beta_n(x, x') < 2.$$

Suppose condition 2 does not hold. Then there exist  $A \in \mathcal{A}^0$  and  $(x, x') \in H$  such that either  $0 < \mathbf{P}^x(A) < 1$  or  $\mathbf{P}^x(A) \neq \mathbf{P}^{x'}(A)$ . Define  $f = \mathbf{1}_A - \mathbf{1}_{A^c}$  and fix  $\alpha > 0$ . Proceeding as in the proof of Theorem 2.1, we find that

$$\begin{aligned} \mathbf{Q}^{x,x'}[|\mathbf{E}^{X_n}[f \circ \Theta^{-n}] - \mathbf{E}^{X'_n}[f \circ \Theta^{-n}]| > 2 - \alpha] &\xrightarrow{n \rightarrow \infty} \\ \mathbf{P}^x(A)\mathbf{P}^{x'}(A^c) + \mathbf{P}^x(A^c)\mathbf{P}^{x'}(A) &> 0. \end{aligned}$$

Note that as  $|f| \leq 1$  and  $f \circ \Theta^{-n}$  is  $\mathcal{A}^0$ -measurable, we have

$$|\mathbf{E}^{X_n}[f \circ \Theta^{-n}] - \mathbf{E}^{X'_n}[f \circ \Theta^{-n}]| \leq \beta(X_n, X'_n) = \beta(x, x') \quad \mathbf{Q}^{x,x'}\text{-a.s.}$$

It follows that  $\beta(x, x') > 2 - \alpha$ , and we therefore have  $\beta(x, x') = 2$  as  $\alpha > 0$  was arbitrary. But by construction there exists  $n \geq 0$  such that  $\beta(x, x') \leq \beta_n(x, x') < 2$ , and we have the desired contradiction.  $\square$

Theorems 2.1 and 2.2, while elementary, play a fundamental role in our theory. In the following subsections, we will see that these results have a broad range of applicability that goes far beyond the setting of Harris chains.

2.2. *Local mixing in infinite dimension.* Markov chains in an infinite-dimensional state space are rarely amenable to the classical theory of Harris chains. The key obstacle is that total variation convergence requires non-singularity of the transition probabilities. This is not restrictive in finite dimension, but fails in infinite dimension even in the most trivial examples.

EXAMPLE 2.3. Let  $(X_k)_{k \geq 0}$  be the Markov chain in  $\{-1, +1\}^{\mathbb{N}}$  such that each coordinate  $(X_k^i)_{k \geq 0}$  is an independent Markov chain in  $\{-1, +1\}$  with transition probabilities  $0 < p_{-1,+1} = p_{+1,-1} < 1/2$ . Clearly each coordinate is a Harris chain, and the law of  $X_n$  converges weakly as  $n \rightarrow \infty$  to its unique invariant measure  $\lambda$  for any initial condition. Nonetheless  $\delta_{1^{\mathbb{N}}} P^n$  and  $\lambda$  are mutually singular for all  $n \geq 0$  ( $\delta_{1^{\mathbb{N}}} P^n$  and  $\lambda$  possess i.i.d. coordinates with a different law), so  $X_n$  cannot converge in total variation.

As the classical measure-theoretic theory fails to yield satisfactory results, the ergodic theory of infinite-dimensional Markov chains is frequently approached by means of topological methods. A connection between topological methods and local zero-two laws will be investigated in section 2.4 below. On the other hand, one may seek a purely measure-theoretic counterpart of the notion of a Harris chain that is adapted to the infinite-dimensional setting. We now describe such a notion due to H. Föllmer [15].

Throughout this section, we adopt the same setting as in section 2.1. To formalize the notion of an infinite-dimensional Markov chain, we assume that the state space  $(E, \mathcal{E})$  is contained in a countable product: that is, there exist a countable set  $I$  and measurable spaces  $(E^i, \mathcal{E}^i)$  such that

$$(E, \mathcal{E}) \subseteq \prod_{i \in I} (E^i, \mathcal{E}^i).$$

Each  $i \in I$  plays the role of a single dimension of the model. We will write  $x = (x^i)_{i \in I}$  for  $x \in E$ , and for  $J \subseteq I$  we denote by  $x^J = (x^i)_{i \in J}$  the natural projection of  $x$  onto  $\prod_{i \in J} E^i$ . For  $m < n$ , we define the quantities  $X_{m,n}^J$  and  $\mathcal{F}_{m,n}^J$  in the obvious manner. Moreover, we define the local tail  $\sigma$ -fields

$$\mathcal{A}^J = \bigcap_{n \geq 0} \mathcal{F}_{n,\infty}^J, \quad \mathcal{A}_{\text{loc}} = \bigvee_{|J| < \infty} \mathcal{A}^J.$$

That is,  $\mathcal{A}_{\text{loc}}$  is generated by the asymptotic events associated to all finite-dimensional projections of the infinite-dimensional chain.

We now introduce Föllmer's notion of local mixing, which states that each finite-dimensional projection of the model converges in total variation. Let us emphasize, as in the previous section, that the finite-dimensional projection of an infinite-dimensional Markov chain is generally not Markov.

DEFINITION 2.4 (Local mixing). A Markov chain  $(X_k)_{k \geq 0}$  taking values in the countable product space  $(E, \mathcal{E}) \subseteq \prod_{i \in I} (E^i, \mathcal{E}^i)$  is *locally mixing* if

$$\|\mathbf{P}^\mu - \mathbf{P}^\nu\|_{\mathcal{F}_{n,\infty}^J} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } \mu, \nu \in \mathcal{P}(E) \text{ and } J \subseteq I, |J| < \infty.$$

In the finite-dimensional case  $|I| < \infty$ , this definition reduces to the ergodic property of Harris chains. Moreover, in the infinite-dimensional setting, Föllmer [15] proves a characterization of local mixing in complete analogy with the Blackwell-Orey equivalence in the theory of Harris chains [34, Ch. 6]. It therefore appears that local mixing is the natural measure-theoretic generalization of the Harris theory to the infinite dimensional setting.

Unfortunately, the characterization given in [15] is of limited use for the purpose of establishing the local mixing property of a given Markov chain: only a very strong verifiable sufficient condition is given there (in the spirit of the Dobrushin uniqueness condition for Gibbs measures). The missing ingredient is a zero-two law, which we can now give as a simple corollary of the results in section 2.1. This completes the characterization of local mixing given in [15], and provides a concrete tool to verify this property.

COROLLARY 2.5 (Local mixing theorem). *The following are equivalent.*

1.  $(X_k)_{k \geq 0}$  is locally mixing.
2.  $\mathcal{A}_{\text{loc}}$  is  $\mathbf{P}^\mu$ -trivial for every  $\mu \in \mathcal{P}(E)$ .
3. For every  $J \subseteq I$ ,  $|J| < \infty$ , there exists  $\alpha > 0$  such that

$$\forall x, x' \in E, \quad \exists n \geq 0 \quad \text{such that} \quad \|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{F}_{n,\infty}^J} \leq 2 - \alpha.$$

PROOF. Note that  $\mathcal{A}_{\text{loc}}$  is  $\mathbf{P}^\mu$ -trivial if and only if  $\mathcal{A}^J$  is  $\mathbf{P}^\mu$ -trivial for all  $|J| < \infty$ . Thus the result follows immediately from Theorem 2.1.  $\square$

Condition 3 of Corollary 2.5 can be used directly to verify the local mixing property in infinite-dimensional models that possess a sufficient degree of nondegeneracy. For example, in the setting of stochastic Navier-Stokes equations with additive noise (cf. section 5.2), the approach developed in [13, 27] can be used to show that Condition 3 holds under the assumption that every Fourier mode is forced by an independent Brownian motion (in this setting, each dimension  $i \in I$  corresponds to a single Fourier mode of the system). However, in degenerate models (for example, where some modes are unforced or when the noise is not additive), local mixing may be difficult or impossible to establish. In section 2.4 below, we will introduce a technique that will significantly extend the applicability of our results.

REMARK 2.6. One can of course also obtain a stationary counterpart of Corollary 2.5 by applying Theorem 2.2 rather than Theorem 2.1. As the result is essentially identical, we do not state it explicitly.

2.3. *Ergodicity of non-Markov processes.* As the local zero-two laws introduced in section 2.1 are essentially non-Markovian, they can be used to investigate the ergodic theory of non-Markov processes. Let us illustrate this idea by developing a new (to the best of our knowledge) characterization of stationary absolutely regular sequences.

In this section, we assume that  $(E, \mathcal{E})$  is a Polish space (the Polish assumption is made to ensure the existence of regular conditional probabilities), and let  $\mathbf{P}$  be a stationary probability measure on  $(\Omega, \mathcal{F})$ . Let us recall the well-known notion of absolute regularity [45] (sometimes called  $\beta$ -mixing).

DEFINITION 2.7 (Absolute regularity). A stationary sequence  $(X_k)_{k \in \mathbb{Z}}$  is said to be *absolutely regular* if the following holds:

$$\|\mathbf{P}[X_{-\infty,0}, X_{k,\infty} \in \cdot] - \mathbf{P}[X_{-\infty,0} \in \cdot] \otimes \mathbf{P}[X_{k,\infty} \in \cdot]\| \xrightarrow{k \rightarrow \infty} 0.$$

We obtain the following characterization.

COROLLARY 2.8. Let  $(X_k)_{k \in \mathbb{Z}}$  be a stationary sequence. Choose any version  $\mathbf{P}^{x_{-\infty,0}}$  of the regular conditional probability  $\mathbf{P}[\cdot | \mathcal{F}_-]$ , and define the measure  $\mathbf{P}^- = \mathbf{P}[X_{-\infty,0} \in \cdot]$ . The following are equivalent:

1.  $(X_k)_{k \in \mathbb{Z}}$  is absolutely regular.
2.  $\|\mathbf{P}^{x_{-\infty,0}} - \mathbf{P}\|_{\mathcal{F}_{k,\infty}} \xrightarrow{k \rightarrow \infty} 0$  for  $\mathbf{P}^-$ -a.e.  $x_{-\infty,0}$ .
3. For  $\mathbf{P}^- \otimes \mathbf{P}^-$ -a.e.  $(x_{-\infty,0}, \tilde{x}_{-\infty,0})$ , there exists  $k \geq 0$  such that  $\mathbf{P}^{x_{-\infty,0}}$  and  $\mathbf{P}^{\tilde{x}_{-\infty,0}}$  are not mutually singular on  $\mathcal{F}_{k,\infty}$ .

PROOF. It is standard (for example, [45]) that  $(X_k)_{k \in \mathbb{Z}}$  is absolutely regular if and only if  $\mathbf{E}[\|\mathbf{P}[X_{k,\infty} \in \cdot | \mathcal{F}_-] - \mathbf{P}[X_{k,\infty} \in \cdot]\|] \rightarrow 0$  as  $k \rightarrow \infty$ . However, as  $\|\mathbf{P}[X_{k,\infty} \in \cdot | \mathcal{F}_-] - \mathbf{P}[X_{k,\infty} \in \cdot]\|$  is pointwise decreasing in  $k$ , the equivalence between conditions 1 and 2 follows immediately.

Now define the  $E^{\mathbb{Z}_-}$ -valued process  $Z_k = X_{-\infty,k}$ . Then  $(Z_k)_{k \in \mathbb{Z}}$  is clearly a Markov chain with transition kernel  $Q(z, A) = \mathbf{P}^z[(z, X_1) \in A]$  and invariant probability  $\mathbf{P}^-$ . We apply Theorem 2.2 to the Markov chain  $(Z_k)_{k \in \mathbb{Z}}$ , where  $\mathcal{E}^0$  in Theorem 2.2 is the  $\sigma$ -field generated by the first coordinate of  $E^{\mathbb{Z}_-}$ . This yields immediately the equivalence between conditions 2 and 3.  $\square$

While conditions 1 and 2 of Corollary 2.8 are standard, condition 3 appears at first sight to be substantially weaker: all that is needed is that, for

almost every pair of initial histories, we can couple the future evolutions with nonzero success probability. This is reminiscent to the corresponding result for Harris chains, and one could argue that absolutely regular sequences provide a natural generalization of the Harris theory to non-Markov processes. In this spirit, Berbee [1] has shown that absolutely regular sequences admit a decomposition into cyclic classes much like in the Markov setting.

The elementary observation used in the proof of Corollary 2.8 is that any non-Markov process  $X_k$  can be made Markov by considering the history process  $Z_k = X_{-\infty, k}$ . However, the process  $Z_k$  is highly degenerate: its transition probabilities are mutually singular for any distinct pair of initial conditions. For this reason, the classical Harris theory is of no use in investigating the ergodicity of non-Markov processes; the *local* nature of the zero-two laws developed in section 2.1 is the key to obtaining nontrivial results.

REMARK 2.9. Along the same lines, one can also obtain a counterpart of Theorem 2.1 for non-Markov processes. The latter is useful for the investigation of delay equations with infinite memory. As no new ideas are involved, we leave the formulation of such a result to the reader.

2.4. *Weak convergence and asymptotic coupling.* In the previous sections, we have employed the local zero-two laws directly to obtain ergodic properties in the total variation distance. However, even local total variation convergence is still too strong a requirement in many cases of interest. In this section, we introduce a technique that allows us to deduce ergodic properties of weak convergence type from the local zero-two laws. This significantly extends the range of applicability of our techniques.

Throughout this section, we adopt the same setting as in section 2.1. We will assume in addition that the state space  $E$  is Polish and is endowed with its Borel  $\sigma$ -field  $\mathcal{E}$  and a complete metric  $d$ . Denote by  $U_b(E)$  the uniformly continuous and bounded functions on  $E$ , and let  $\text{Lip}(E)$  be the class of functions  $f \in U_b(E)$  such that  $\|f\|_\infty \leq 1$  and  $|f(x) - f(y)| \leq d(x, y)$  for all  $x, y \in E$ . Let  $\mathcal{M}(E)$  be the space of signed finite measures on  $E$ , and define the bounded-Lipschitz norm  $\|\varrho\|_{\text{BL}} = \sup_{f \in \text{Lip}(E)} |\varrho f|$  for  $\varrho \in \mathcal{M}(E)$ . We recall for future reference that  $x \mapsto \|K(x, \cdot) - K'(x, \cdot)\|_{\text{BL}}$  is measurable when  $K, K'$  are finite kernels; see, for example, [41, Lemma A.1].

A *coupling* of two probability measures  $\mathbf{P}_1, \mathbf{P}_2$  on  $\Omega$  is a probability measure  $\mathbf{Q}$  on  $\Omega \times \Omega$  such that the first marginal of  $\mathbf{Q}$  coincides with  $\mathbf{P}_1$  and the second marginal coincides with  $\mathbf{P}_2$ . Let us denote the family of all couplings of  $\mathbf{P}_1, \mathbf{P}_2$  by  $\mathcal{C}(\mathbf{P}_1, \mathbf{P}_2)$ . To set the stage for our result, let us recall

the coupling characterization of the total variation distance:

$$\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathcal{F}_{n,\infty}} = 2 \min\{\mathbf{Q}[X_{n,\infty} \neq X'_{n,\infty}] : \mathbf{Q} \in \mathcal{C}(\mathbf{P}_1, \mathbf{P}_2)\}.$$

Consider for simplicity the classical zero-two law (Theorem 2.1 for  $\mathcal{E}^0 = \mathcal{E}$ ). Its basic condition reads: there exists  $\alpha > 0$  such that

$$\forall x, x' \in E \quad \exists n \geq 0 \quad \text{such that} \quad \|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{F}_{n,\infty}} \leq 2 - \alpha.$$

By the coupling characterization of the total variation distance, this condition can be equivalently stated as follows: there exists  $\alpha > 0$  such that

$$\forall x, x' \in E \quad \exists \mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'}) \quad \text{such that} \quad \mathbf{Q}\left[\sum_{n=0}^{\infty} \mathbf{1}_{X_n \neq X'_n} < \infty\right] \geq \alpha.$$

The message of the following theorem is that if one replaces the discrete distance  $\mathbf{1}_{X_n \neq X'_n}$  by the topological distance  $d(X_n, X'_n)$ , one obtains an ergodic theorem with respect to the bounded-Lipschitz (rather than total variation) distance. This is a much weaker assumption: it is not necessary to construct an *exact* coupling where  $X_n, X'_n$  eventually coincide with positive probability, but only an *asymptotic* coupling where  $X_n, X'_n$  converge towards each other. The latter can often be accomplished even in degenerate situations.

**THEOREM 2.10** (Weak-\* ergodicity). *Suppose there exists  $\alpha > 0$  so that*

$$\forall x, x' \in E \quad \exists \mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'}) \quad \text{such that} \quad \mathbf{Q}\left[\sum_{n=0}^{\infty} d(X_n, X'_n)^2 < \infty\right] \geq \alpha.$$

*Then the Markov chain is weak-\* ergodic in the sense that*

$$\|\mu P^n - \nu P^n\|_{\text{BL}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for every } \mu, \nu \in \mathcal{P}(E).$$

It is interesting to compare Theorem 2.10 to the weak-\* ergodic theorems obtained in [17, section 2.2] in terms of asymptotic coupling. In contrast to those results, Theorem 2.10 requires no specific recurrence structure, Markovian couplings, or control on the coupling probability  $\alpha$  as a function of  $x, x'$ . On the other hand, Theorem 2.10 requires the asymptotic coupling to converge sufficiently quickly so that  $\sum d(X_n, X'_n)^2 < \infty$  (this is not a serious issue in most applications), while the results in [17] are in principle applicable to couplings with an arbitrarily slow convergence rate. These results therefore provide complementary conditions for weak-\* ergodicity.

However, it should be emphasized that the feature of Theorem 2.10 that is of key importance for our purposes is that its proof reduces the problem to

the local zero-two law of section 2.1. Using this technique, we can therefore extend the applicability of purely measure-theoretic results that are based on zero-two laws to a wide class of weak-\* ergodic Markov chains. This idea will be crucial to establishing conditional ergodicity in degenerate infinite-dimensional models (see section 5 for examples).

**PROOF OF THEOREM 2.10.** Let  $(\bar{E}, \bar{\mathcal{E}}) = (E \times \mathbb{R}, \mathcal{E} \otimes \mathcal{B}(\mathbb{R}))$ . Consider the  $\bar{E}$ -valued process  $(Z_n)_{n \geq 0}$  (defined on its canonical probability space) such that  $Z_n = (X_n, \xi_n)$ , where  $(\xi_n)_{n \geq 0}$  is an i.i.d. sequence of standard Gaussian random variables independent of the Markov chain  $(X_n)_{n \geq 0}$ . Clearly  $(Z_n)_{n \geq 0}$  is itself a Markov chain. Given  $f \in \text{Lip}(E)$ , we will apply Theorem 2.1 to  $(Z_n)_{n \geq 0}$  with the local  $\sigma$ -field  $\bar{\mathcal{E}}^0 = \sigma\{g\}$ ,  $g(x, y) = f(x) + y$ .

We begin by noting a standard estimate.

**LEMMA 2.11.** *Let  $(\xi_n)_{n \geq 0}$  be an i.i.d. sequence of standard Gaussian random variables, and let  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  be real-valued sequences. Then*

$$\|\mathbf{P}[(a_n + \xi_n)_{n \geq 0} \in \cdot] - \mathbf{P}[(b_n + \xi_n)_{n \geq 0} \in \cdot]\|^2 \leq \sum_{n=0}^{\infty} (b_n - a_n)^2.$$

**PROOF.** Denote by  $H(\mu, \nu) = \int \sqrt{d\mu d\nu}$  the Kakutani-Hellinger affinity between probability measures  $\mu, \nu$ . We recall that [36, section III.9]

$$\|\mu_1 \otimes \cdots \otimes \mu_n - \nu_1 \otimes \cdots \otimes \nu_n\|^2 \leq 8 \left[ 1 - \prod_{k=1}^n H(\mu_k, \nu_k) \right].$$

But a direct computation shows that  $H(N(a, 1), N(b, 1)) = \exp(-(b-a)^2/8)$ . The result now follows directly using  $1 - e^{-x} \leq x$  and  $n \rightarrow \infty$ .  $\square$

Fix  $x, x' \in E$  and  $f \in \text{Lip}(E)$ , and choose  $\mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'})$  as in the statement of the theorem. By assumption, we can choose  $n \geq 0$  such that

$$\mathbf{Q} \left[ \sum_{k=n}^{\infty} d(X_k, X'_k)^2 \leq \frac{\alpha^2}{4} \right] \geq \frac{3\alpha}{4}.$$

Let  $F_n = f(X_n) + \xi_n$ , and define for every real-valued sequence  $\mathbf{a} = (a_n)_{n \geq 0}$  the measure  $\mu_{\mathbf{a}} = \mathbf{P}[(a_n + \xi_n)_{n \geq 0} \in \cdot]$ . Then we have for every  $A \in \mathcal{B}(\mathbb{R})^{\mathbb{Z}_+}$

$$\mathbf{P}^x[F_{n,\infty} \in A] - \mathbf{P}^{x'}[F_{n,\infty} \in A] = \mathbf{E}_{\mathbf{Q}}[\mu_{(f(X_k))_{k \geq n}}(A) - \mu_{(f(X'_k))_{k \geq n}}(A)].$$



Therefore, we obtain by Jensen's inequality and Lemma 2.11

$$\begin{aligned} \|\mathbf{P}^x[F_{n,\infty} \in \cdot] - \mathbf{P}^{x'}[F_{n,\infty} \in \cdot]\| &\leq \mathbf{E}_{\mathbf{Q}} \left[ \left( \sum_{k=n}^{\infty} \{f(X_k) - f(X'_k)\}^2 \right)^{1/2} \wedge 2 \right] \\ &\leq \mathbf{E}_{\mathbf{Q}} \left[ \left( \sum_{k=n}^{\infty} d(X_k, X'_k)^2 \right)^{1/2} \wedge 2 \right] \\ &\leq \frac{\alpha}{2} + 2 \left[ 1 - \frac{3\alpha}{4} \right] = 2 - \alpha, \end{aligned}$$

where we have used the Lipschitz property of  $f$ . Applying Theorem 2.1 as indicated at the beginning of the proof, it follows that

$$\|\mathbf{P}^\mu[F_{n,\infty} \in \cdot] - \mathbf{P}^\nu[F_{n,\infty} \in \cdot]\| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } \mu, \nu \in \mathcal{P}(E).$$

In particular, if we denote by  $\xi \in \mathcal{P}(\mathbb{R})$  the standard Gaussian measure, then

$$\|\mu P^n f^{-1} * \xi - \nu P^n f^{-1} * \xi\| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } \mu, \nu \in \mathcal{P}(E)$$

(here  $*$  denotes convolution). We claim that this implies

$$|\mu P^n f - \nu P^n f| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } \mu, \nu \in \mathcal{P}(E) \text{ and } f \in \text{Lip}(E).$$

Indeed, if we assume the contrary, then there exists for some  $f \in \text{Lip}(E)$  and  $\mu, \nu \in \mathcal{P}(E)$  a subsequence  $m_n \uparrow \infty$  so that  $\inf_n |\mu P^{m_n} f - \nu P^{m_n} f| > 0$ . As  $f$  takes values in the compact interval  $[-1, 1]$ , we can extract a further subsequence  $k_n \uparrow \infty$  so that  $\mu P^{k_n} f^{-1} \rightarrow \mu_\infty$  and  $\nu P^{k_n} f^{-1} \rightarrow \nu_\infty$  in the weak convergence topology for some  $\mu_\infty, \nu_\infty \in \mathcal{P}([-1, 1])$ , and clearly  $\mu_\infty \neq \nu_\infty$  by construction. On the other hand, as  $\|\mu P^n f^{-1} * \xi - \nu P^n f^{-1} * \xi\| \rightarrow 0$ , we must have  $\mu_\infty * \xi = \nu_\infty * \xi$ . This entails a contradiction, as the Fourier transform of  $\xi$  vanishes nowhere (so convolution by  $\xi$  is injective on  $\mathcal{P}(\mathbb{R})$ ).

We finally claim that in fact

$$\|\mu P^n - \nu P^n\|_{\text{BL}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } \mu, \nu \in \mathcal{P}(E).$$

Indeed, we have shown above that the signed measure  $\varrho_n = \mu P^n - \nu P^n$  converges to zero pointwise on  $\text{Lip}(E)$ . As any function in  $U_b(E)$  can be approximated uniformly by bounded Lipschitz functions, this implies that  $\varrho_n \rightarrow 0$  in the  $\sigma(\mathcal{M}(E), U_b(E))$ -topology. A result of Pachl [33, Theorem 3.2] now implies that  $\|\varrho_n\|_{\text{BL}} \rightarrow 0$ , which concludes the proof of the theorem.  $\square$

For simplicity, we have stated the assumption of Theorem 2.10 so that an asymptotic coupling of the entire state  $X_k$  of the Markov chain is required. The reader may easily adapt the proof of Theorem 2.10 to require only asymptotic couplings of finite-dimensional projections as in the local mixing setting (Corollary 2.5), or to deduce a variant of this result in the setting of non-Markov processes. However, let us emphasize that even in the setting of Theorem 2.10, where the asymptotic coupling is at the level of the Markov process  $X_k$ , the “smoothed” process  $F_k = f(X_k) + \xi_k$  that appears in the proof is non-Markovian. Therefore, the *local* zero-two law is essential in order to obtain weak-\* ergodicity results from the total variation theory.

The stationary counterpart to Theorem 2.10 also follows along the same lines. However, here a small modification is needed at the end of the proof.

**THEOREM 2.12** (Stationary weak-\* ergodicity). *Let  $\lambda$  be a  $P$ -invariant probability. Suppose that for  $\lambda \otimes \lambda$ -a.e.  $(x, x') \in E \times E$ ,*

$$\exists \mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'}) \quad \text{such that} \quad \mathbf{Q} \left[ \sum_{n=0}^{\infty} d(X_n, X'_n)^2 < \infty \right] > 0.$$

*Then the Markov chain is a.e. weak-\* ergodic in the sense that*

$$\|P^n(x, \cdot) - \lambda\|_{\text{BL}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \lambda\text{-a.e. } x \in E.$$

**PROOF.** Repeating the proof of Theorem 2.10 using Theorem 2.2 instead of Theorem 2.1 yields the following: for every  $f \in \text{Lip}(E)$ , we have

$$|P^n f(x) - \lambda f| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \lambda\text{-a.e. } x.$$

We would like to extend this to convergence in the bounded-Lipschitz norm. This does not follow immediately, however, as the  $\lambda$ -null set of  $x \in E$  for which the convergence fails may depend on  $f \in \text{Lip}(E)$ .

Fix  $\varepsilon > 0$ . Let  $K \subseteq E$  be a compact set such that  $\lambda(K) \geq 1 - \varepsilon$ , and define  $\chi(x) = (1 - \varepsilon^{-1}d(x, K))_+$ . Then we can estimate

$$\begin{aligned} \|P^n(x, \cdot) - \lambda\|_{\text{BL}} &\leq \sup_{f \in \text{Lip}(E)} |P^n(f\chi)(x) - \lambda(f\chi)| + P^n(1 - \chi)(x) + \lambda(1 - \chi) \\ &\leq \sup_{f \in \text{Lip}(E)} |P^n(f\chi)(x) - \lambda(f\chi)| + |P^n\chi(x) - \lambda\chi| + 2\varepsilon. \end{aligned}$$

By the Arzelà-Ascoli theorem, we can find a finite number of functions  $f_1, \dots, f_k \in \text{Lip}(E)$  such that  $\sup_{f \in \text{Lip}(E)} \min_i \|f_i \mathbf{1}_K - f \mathbf{1}_K\|_{\infty} \leq \varepsilon$ . But

note that  $|f(x) - g(x)| \leq 2\varepsilon + \|f\mathbf{1}_K - g\mathbf{1}_K\|_\infty$  whenever  $d(x, K) \leq \varepsilon$  and  $f, g \in \text{Lip}(E)$ . Therefore,  $\sup_{f \in \text{Lip}(E)} \min_i \|f_i\chi - f\chi\|_\infty \leq 3\varepsilon$ , and we have

$$\|P^n(x, \cdot) - \lambda\|_{\text{BL}} \leq \max_{i=1, \dots, k} |P^n(f_i\chi)(x) - \lambda(f_i\chi)| + |P^n\chi(x) - \lambda\chi| + 8\varepsilon.$$

As the quantity on the right-hand side depends only on a finite number of bounded Lipschitz functions  $\chi, f_1\chi, \dots, f_k\chi$ , we certainly have

$$\limsup_{n \rightarrow \infty} \|P^n(x, \cdot) - \lambda\|_{\text{BL}} \leq 8\varepsilon \quad \text{for } \lambda\text{-a.e. } x.$$

But  $\varepsilon > 0$  was arbitrary, so the proof is complete.  $\square$

**REMARK 2.13.** The tightness argument used here is in fact more elementary than the result of Pachl [33] used in the proof of Theorem 2.10. Note, however, that Theorem 2.10 does not even require the existence of an invariant probability, so that tightness is not guaranteed in that setting.

**3. Conditional ergodicity.** In the previous section, we have developed various measure-theoretic ergodic theorems that are applicable in infinite-dimensional or non-Markov settings. The goal of the present section is to develop a *conditional* variant of these ideas: given a stationary process  $(Z_k, Y_k)_{k \in \mathbb{Z}}$ , we aim to understand when  $(Z_k)_{k \in \mathbb{Z}}$  is ergodic conditionally on  $(Y_k)_{k \in \mathbb{Z}}$ . The conditional ergodic theory developed in this section will be used in section 4 below to prove stability and ergodicity of nonlinear filters.

In section 3.1, we first develop a conditional variant of the zero-two laws of the previous section. In principle, this result completely characterizes the conditional absolute regularity property of  $(Z_k)_{k \in \mathbb{Z}}$  given  $(Y_k)_{k \in \mathbb{Z}}$ . Unfortunately, the equivalent conditions of the zero-two law are stated in terms of the conditional distribution  $\mathbf{P}[Z \in \cdot | \mathcal{F}^Y]$ : this quantity is defined abstractly as a regular conditional probability, but an explicit expression is almost never available. Therefore, in itself, the conditional zero-two law is very difficult to use. In contrast, the (unconditional) ergodic theory of  $(Z_k, Y_k)_{k \in \mathbb{Z}}$  can typically be studied by direct analysis of the underlying model.

The question that we aim to address is therefore that of inheritance: if the unconditional process  $(Z_k, Y_k)_{k \in \mathbb{Z}}$  is absolutely regular, does this imply that the process is also conditionally absolutely regular? In general this is not the case even when the process is Markov (cf. [43]). However, under a suitable *nondegeneracy* requirement, we will be able to establish inheritance of the absolute regularity property from the unconditional to the conditional process (section 3.2). Using an additional argument (section 3.3), we will also deduce conditional ergodicity given the one-sided process  $(Y_k)_{k \geq 0}$  rather than the two-sided process  $(Y_k)_{k \in \mathbb{Z}}$ , as will be needed in section 4 below.

The inheritance of the ergodicity property under conditioning was first established in the Markov setting in [40, 39]. For a Markov process  $(X_k)_{k \geq 0}$ , the condition of the zero-two law states that for a.e. initial conditions  $x, x'$ , there exists  $n \geq 0$  such that  $P^n(x, \cdot)$  and  $P^n(x', \cdot)$  are not mutually singular, while the conditional zero-two law yields essentially the same condition where the transition kernel  $P(x, \cdot) = \mathbf{P}^x[X_1 \in \cdot]$  is replaced by the conditional transition kernel  $\tilde{P}(x, \cdot) = \mathbf{P}^x[X_1 \in \cdot | \mathcal{F}^Y]$ . The key idea in the proof of inheritance was to show that the unconditional and conditional transition kernels are equivalent  $P(x, \cdot) \sim \tilde{P}(x, \cdot)$  a.e., which immediately yields equivalence of the conditions of the unconditional and conditional zero-two laws. Unfortunately, such an approach cannot work in the non-Markov setting, as here the corresponding argument would require us to show the equivalence of laws of the infinite future  $\mathbf{P}[Z_{k,\infty} \in \cdot | \mathcal{F}_{-\infty,0}^Z]$  and  $\mathbf{P}[Z_{k,\infty} \in \cdot | \mathcal{F}^Y \vee \mathcal{F}_{-\infty,0}^Z]$ . Such an equivalence on the infinite time interval cannot hold except in trivial cases (even in the Markov setting). Instead, we develop a new method to establish inheritance of the conditions of the unconditional and conditional zero-two laws that avoids the Markov-specific argument used in [40, 39].

Throughout this section, we adopt the same setting and notations as in section 2. Here we will assume that  $E = G \times F$  where  $G, F$  are Polish spaces, and we fix a stationary probability  $\mathbf{P}$  on  $(\Omega, \mathcal{F})$ . We denote the components of the coordinate process as  $X_n = (Z_n, Y_n)$ . Thus  $(Z_n, Y_n)_{n \in \mathbb{Z}}$  is a stationary process in  $G \times F$  defined on the canonical probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Let

$$\mathcal{F}_{m,n}^Z = \sigma\{Z_{m,n}\}, \quad \mathcal{F}_{m,n}^Y = \sigma\{Y_{m,n}\}, \quad \mathcal{F}^Z = \mathcal{F}_{-\infty,\infty}^Z, \quad \mathcal{F}^Y = \mathcal{F}_{-\infty,\infty}^Y,$$

so that  $\mathcal{F}_{m,n} = \mathcal{F}_{m,n}^Z \vee \mathcal{F}_{m,n}^Y$ . We also define the tail  $\sigma$ -field

$$\mathcal{A}^Z = \bigcap_{n \geq 0} \mathcal{F}_{n,\infty}^Z.$$

For simplicity, we will write  $Z = Z_{-\infty,\infty}$  and  $Y = Y_{-\infty,\infty}$ . We introduce the convention that for a sequence  $z = (z_n)_{n \in \mathbb{Z}}$ , we write  $z_- = (z_n)_{n \leq 0}$ .

**3.1. A conditional zero-two law.** In this section, we establish a conditional counterpart to Corollary 2.8 that characterizes the absolute regularity property of  $(Z_n)_{n \in \mathbb{Z}}$  conditionally on  $(Y_n)_{n \in \mathbb{Z}}$ . While it is difficult to apply directly, this conditional zero-two law plays a fundamental role in the study of conditional ergodicity to be undertaken in the following sections.

In the following, we define the probability measure  $\mathbf{Y}$  and fix versions of the regular conditional probabilities  $\mathbf{P}_y, \mathbf{P}_y^{z_-}$  as follows:

$$\mathbf{Y} = \mathbf{P}[Y \in \cdot], \quad \mathbf{P}_Y = \mathbf{P}[Z \in \cdot | \mathcal{F}^Y], \quad \mathbf{P}_Y^{z_-} = \mathbf{P}[Z \in \cdot | \mathcal{F}^Y \vee \mathcal{F}_{-\infty,0}^Z].$$

The following is the main result of this section.

**THEOREM 3.1** (Conditional 0-2 law). *The following are equivalent:*

1. For  $(\mathbf{P}_y \otimes \mathbf{P}_y)\mathbf{Y}$ -a.e.  $(z, z', y)$ , we have

$$\|\mathbf{P}_y^{z-} - \mathbf{P}_y^{z'-}\|_{\mathcal{F}_{n,\infty}^Z} \xrightarrow{n \rightarrow \infty} 0.$$

2. For  $(\mathbf{P}_y \otimes \mathbf{P}_y)\mathbf{Y}$ -a.e.  $(z, z', y)$ , we have

$$\mathbf{P}_y^{z-}(A) = \mathbf{P}_y^{z-}(A)^2 = \mathbf{P}_y^{z'-}(A) \quad \text{for all } A \in \mathcal{A}^Z.$$

3. For  $(\mathbf{P}_y \otimes \mathbf{P}_y)\mathbf{Y}$ -a.e.  $(z, z', y)$ , there exists  $n \geq 0$  such that  $\mathbf{P}_y^{z-}$  and  $\mathbf{P}_y^{z'-}$  are not mutually singular on  $\mathcal{F}_{n,\infty}^Z$ .

The proof of Theorem 3.1 is similar in spirit to that of Theorem 2.2. However, some care is needed in the handling of regular conditional probabilities.

We begin by establishing the following stationarity result.

**LEMMA 3.2.** *For  $\mathbf{P}$ -a.e.  $(z, y)$ , we have*

$$\mathbf{P}_{\Theta^n y}^{Z-\circ\Theta^n}(A) = \mathbf{E}_y^{z-}[\mathbf{1}_A \circ \Theta^n | \mathcal{F}_{-\infty,n}^Z] \quad \mathbf{P}_y^{z-}\text{-a.s.} \quad \text{for every } A \in \mathcal{F}^Z.$$

**PROOF.** Fix  $A \in \mathcal{F}^Z$ ,  $B \in \mathcal{F}_{-\infty,n}^Z$ , and  $C \in \mathcal{F}^Y \vee \mathcal{F}_{-\infty,0}^Z$ . Then

$$\begin{aligned} \mathbf{E}\left[\mathbf{E}_Y^{Z-}\left[\mathbf{P}_{\Theta^n Y}^{Z-\circ\Theta^n}(A) \mathbf{1}_B\right] \mathbf{1}_C\right] &= \mathbf{E}\left[\mathbf{P}_{\Theta^n Y}^{Z-\circ\Theta^n}(A) \mathbf{1}_{B \cap C}\right] \\ &= \mathbf{E}\left[\mathbf{P}_Y^{Z-}(A) \{\mathbf{1}_{B \cap C} \circ \Theta^{-n}\}\right] \\ &= \mathbf{E}\left[\mathbf{1}_A \{\mathbf{1}_{B \cap C} \circ \Theta^{-n}\}\right] \\ &= \mathbf{E}\left[\{\mathbf{1}_A \circ \Theta^n\} \mathbf{1}_{B \cap C}\right] \\ &= \mathbf{E}\left[\mathbf{E}_Y^{Z-}[\{\mathbf{1}_A \circ \Theta^n\} \mathbf{1}_B] \mathbf{1}_C\right], \end{aligned}$$

where we have used the stationarity of  $\mathbf{P}$  and the definition of the regular conditional probability  $\mathbf{P}_y^{z-}$ . As this holds for all  $C \in \mathcal{F}^Y \vee \mathcal{F}_{-\infty,0}^Z$ , we have

$$\mathbf{E}_y^{z-}\left[\mathbf{P}_{\Theta^n y}^{Z-\circ\Theta^n}(A) \mathbf{1}_B\right] = \mathbf{E}_y^{z-}[\{\mathbf{1}_A \circ \Theta^n\} \mathbf{1}_B]$$

for  $\mathbf{P}$ -a.e.  $(z, y)$ . But as  $\mathcal{F}^Z$  and  $\mathcal{F}_{-\infty,n}^Z$  are countably generated, we can ensure that for  $\mathbf{P}$ -a.e.  $(z, y)$ , this equality holds simultaneously for all sets  $A$  and  $B$  in a countable generating algebra for  $\mathcal{F}^Z$  and  $\mathcal{F}_{-\infty,n}^Z$ , respectively. By the monotone class theorem, it follows that for  $\mathbf{P}$ -a.e.  $(z, y)$  the equality holds for all  $A \in \mathcal{F}^Z$  and  $B \in \mathcal{F}_{-\infty,n}^Z$  simultaneously, which yields the claim.  $\square$

In the following, we define

$$\beta_n(z_-, z'_-, y) = \|\mathbf{P}_y^{z_-} - \mathbf{P}_y^{z'_-}\|_{\mathcal{F}_{n,\infty}^Z}, \quad \beta(z_-, z'_-, y) = \lim_{n \rightarrow \infty} \beta_n(z_-, z'_-, y).$$

The limit in the definition of  $\beta$  exists pointwise as  $\beta_n$  is decreasing in  $n$ . As  $\mathcal{F}_{n,\infty}^Z$  is countably generated, the maps  $\beta_n$  and  $\beta$  are measurable.

Define the probability  $\mathbf{Q} = (\mathbf{P}_y \otimes \mathbf{P}_y)\mathbf{Y}$  on  $G^{\mathbb{Z}} \times G^{\mathbb{Z}} \times F^{\mathbb{Z}}$ , that is,

$$\int \mathbf{1}_A(z, z', y) \mathbf{Q}(dz, dz', dy) = \int \mathbf{1}_A(z, z', y) \mathbf{P}_y(dz) \mathbf{P}_y(dz') \mathbf{Y}(dy).$$

Denote the coordinate process on  $G^{\mathbb{Z}} \times G^{\mathbb{Z}} \times F^{\mathbb{Z}}$  as  $(Z_n, Z'_n, Y_n)_{n \in \mathbb{Z}}$ . Evidently  $\mathbf{Q}$  is a coupling of two copies of  $\mathbf{P}$  such that the observations  $Y$  coincide and  $Z, Z'$  are conditionally independent given  $Y$ . Moreover, as  $\mathbf{Y}$  is stationary and  $\mathbf{P}_{Y \circ \Theta}(A) = \mathbf{P}[A | \mathcal{F}^Y] \circ \Theta = \mathbf{E}[\mathbf{1}_A \circ \Theta | \mathcal{F}^Y] = \mathbf{E}_Y(\mathbf{1}_A \circ \Theta)$   $\mathbf{P}$ -a.s., it is easily seen that  $\mathbf{Q}$  is also a stationary measure. Finally, define

$$\mathbf{Q}_y^{z_-, z'_-} = \mathbf{P}_y^{z_-} \otimes \mathbf{P}_y^{z'_-} \otimes \delta_y \quad \text{on} \quad G^{\mathbb{Z}} \times G^{\mathbb{Z}} \times F^{\mathbb{Z}}.$$

It follows directly that  $\mathbf{Q}_Y^{Z_-, Z'_-}$  is a version of the regular conditional probability  $\mathbf{Q}[\cdot | \mathcal{F}^Y \vee \mathcal{F}_{-\infty,0}^Z \vee \mathcal{F}_{-\infty,0}^{Z'}]$ , where  $\mathcal{F}_{m,n}^{Z'} = \sigma\{Z'_{m,n}\}$ .

The following lemma establishes the invariance of  $\beta$ .

LEMMA 3.3. *For  $\mathbf{Q}$ -a.e.  $(z_-, z'_-, y)$  we have*

$$\mathbf{Q}_y^{z_-, z'_-}[\beta(z_-, z'_-, y) = \beta(Z_- \circ \Theta^n, Z'_- \circ \Theta^n, \Theta^n y) \text{ for all } n \geq 0] = 1.$$

PROOF. Define for simplicity  $\mathcal{G}_n = \mathcal{F}^Y \vee \mathcal{F}_{-\infty,n}^Z \vee \mathcal{F}_{-\infty,n}^{Z'}$ . By Jensen's inequality and conditional independence, we obtain

$$\begin{aligned} \beta_k(Z_-, Z'_-, Y) &= \|\mathbf{Q}[Z_{k,\infty} \in \cdot | \mathcal{G}_0] - \mathbf{Q}[Z'_{k,\infty} \in \cdot | \mathcal{G}_0]\| \\ &\leq \mathbf{E}_{\mathbf{Q}}[\|\mathbf{Q}[Z_{k,\infty} \in \cdot | \mathcal{G}_1] - \mathbf{Q}[Z'_{k,\infty} \in \cdot | \mathcal{G}_1]\| | \mathcal{G}_0] \\ &= \mathbf{E}_{\mathbf{Q}}[\|\mathbf{Q}[Z_{k-1,\infty} \in \cdot | \mathcal{G}_0] - \mathbf{Q}[Z'_{k-1,\infty} \in \cdot | \mathcal{G}_0]\| \circ \Theta | \mathcal{G}_0] \\ &= \mathbf{E}_{\mathbf{Q}}[\beta_{k-1}(Z_-, Z'_-, Y) \circ \Theta | \mathcal{G}_0]. \end{aligned}$$

Letting  $k \rightarrow \infty$  and using stationarity, we find that  $M_n = \beta(Z_-, Z'_-, Y) \circ \Theta^n$  is a bounded submartingale under  $\mathbf{Q}$ . Using stationarity, we find that

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}}[\|\beta(Z_-, Z'_-, Y) - \beta(Z_-, Z'_-, Y) \circ \Theta^n\|] &= \\ \mathbf{E}_{\mathbf{Q}}[\|\beta(Z_-, Z'_-, Y) \circ \Theta^k - \beta(Z_-, Z'_-, Y) \circ \Theta^{n+k}\|] &\xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

The result now follows by disintegration.  $\square$

We are now ready to complete the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. The proofs of  $2 \Rightarrow 1 \Rightarrow 3$  are identical to the corresponding proofs in Theorem 2.1. It remains to prove  $3 \Rightarrow 2$ .

Let us assume that condition 3 holds. Then, using Lemmas 3.2 and 3.3, we can find a measurable subset  $H \subset G^{\mathbb{Z}} \times G^{\mathbb{Z}} \times F^{\mathbb{Z}}$  of full probability  $\mathbf{Q}(H) = 1$  such that the following hold for every  $(z, z', y) \in H$ :

- a.  $\mathbf{P}_y^{z-}$  and  $\mathbf{P}_y^{z'-}$  are not mutually singular on  $\mathcal{F}_{n,\infty}^Z$  for some  $n \geq 0$ .
- b.  $\mathbf{P}_{\Theta^n y}^{Z-\circ\Theta^n}(A) = \mathbf{E}_y^{z-}[\mathbf{1}_A \circ \Theta^n | \mathcal{F}_{-\infty,n}^Z] \mathbf{P}_y^{z-}$ -a.s. for all  $A \in \mathcal{F}^Z$ ,  $n \geq 0$ .
- c.  $\mathbf{P}_{\Theta^n y}^{Z-\circ\Theta^n}(A) = \mathbf{E}_y^{z'-}[\mathbf{1}_A \circ \Theta^n | \mathcal{F}_{-\infty,n}^Z] \mathbf{P}_y^{z'-}$ -a.s. for all  $A \in \mathcal{F}^Z$ ,  $n \geq 0$ .
- d.  $\mathbf{Q}_y^{z-,z'-}[\beta(z_-, z'_-, y) = \beta(Z_- \circ \Theta^n, Z'_- \circ \Theta^n, \Theta^n y) \text{ for all } n \geq 0] = 1$ .

Now suppose that condition 2 does not hold. Then we can choose a path  $(z, z', y) \in H$  and a tail event  $A \in \mathcal{A}^Z$  such that

$$\text{either } 0 < \mathbf{P}_y^{z-}(A) < 1, \quad \text{or} \quad \mathbf{P}_y^{z-}(A) \neq \mathbf{P}_y^{z'-}(A).$$

Define  $f = \mathbf{1}_A - \mathbf{1}_{A^c}$  and fix  $\alpha > 0$ . By the martingale convergence theorem,

$$\begin{aligned} \mathbf{Q}_y^{z-,z'-} \left[ \left| \mathbf{E}_{\Theta^n y}^{Z-\circ\Theta^n}(f \circ \Theta^{-n}) - \mathbf{E}_{\Theta^n y}^{Z'-\circ\Theta^n}(f \circ \Theta^{-n}) \right| > 2 - \alpha \right] &\xrightarrow{n \rightarrow \infty} \\ \mathbf{P}_y^{z-}(A) \mathbf{P}_y^{z'-}(A^c) + \mathbf{P}_y^{z-}(A^c) \mathbf{P}_y^{z'-}(A) &> 0. \end{aligned}$$

Note that as  $|f| \leq 1$  and  $f \circ \Theta^{-n}$  is  $\mathcal{A}^Z$ -measurable, we have

$$\begin{aligned} \left| \mathbf{E}_{\Theta^n y}^{Z-\circ\Theta^n}(f \circ \Theta^{-n}) - \mathbf{E}_{\Theta^n y}^{Z'-\circ\Theta^n}(f \circ \Theta^{-n}) \right| &\leq \beta(Z_- \circ \Theta^n, Z'_- \circ \Theta^n, \Theta^n y) \\ &= \beta(z_-, z'_-, y) \quad \mathbf{Q}_y^{z-,z'-}\text{-a.s.} \end{aligned}$$

It follows that  $\beta(z_-, z'_-, y) > 2 - \alpha$ , and we therefore have  $\beta(z_-, z'_-, y) = 2$  as  $\alpha > 0$  was arbitrary. But by construction there exists  $n \geq 0$  such that  $\beta(z_-, z'_-, y) \leq \beta_n(z_-, z'_-, y) < 2$ , which entails the desired contradiction.  $\square$

**3.2. Nondegeneracy and inheritance.** While Theorem 3.1 in principle characterizes completely the conditional absolute regularity property, this result is difficult to apply directly as an explicit description of  $\mathbf{P}_y^{z-}$  (beyond its existence as a regular conditional probability) is typically not available. On the other hand, many methods are available to establish the absolute regularity property of the unconditional model  $(Z, Y)$ . We will presently develop a technique that allows us to deduce the conditional absolute regularity property from absolute regularity of the unconditional model.

The essential assumption that will be needed for inheritance of the absolute regularity property is nondegeneracy of the observations. Roughly speaking, this requirement ensures that we cannot infer with certainty the outcome of any unobserved event in  $\mathcal{F}^Z$  given a finite number of observations  $Y_m, \dots, Y_n$ . The nondegeneracy assumption is typically easy to verify in practice from the model description (see section 4, for example).

**DEFINITION 3.4** (Nondegeneracy). The process  $(Z_k, Y_k)_{k \in \mathbb{Z}}$  is said to be *nondegenerate* if for every  $-\infty < m \leq n < \infty$  we have

$$\mathbf{P}[Y_{m,n} \in \cdot | \mathcal{F}_{-\infty, m-1} \vee \mathcal{F}_{n+1, \infty}] \sim \mathbf{P}[Y_{m,n} \in \cdot | \mathcal{F}_{-\infty, m-1}^Y \vee \mathcal{F}_{n+1, \infty}^Y] \quad \mathbf{P}\text{-a.s.}$$

We now state the main result of this section (recall that the definition of absolute regularity was given as Definition 2.7 above).

**THEOREM 3.5** (Inheritance of absolute regularity). *Suppose that the stationary process  $(Z_k, Y_k)_{k \in \mathbb{Z}}$  is absolutely regular and nondegenerate. Then any (hence all) of the conditions of Theorem 3.1 hold true.*

To prove this result, we need three lemmas. The first lemma is a basic result on the equivalence of regular conditional probabilities.

**LEMMA 3.6.** *Let  $H_1, H_2, H_3$  be Polish spaces, and let  $\mathbf{R}$  be a probability on  $H_1 \times H_2 \times H_3$ . Define  $X_i(x_1, x_2, x_3) = x_i$  and  $\mathcal{G}_i = \sigma\{X_i\}$ . Then*

$$\mathbf{R}[X_1 \in \cdot | \mathcal{G}_2 \vee \mathcal{G}_3] \sim \mathbf{R}[X_1 \in \cdot | \mathcal{G}_3] \quad \mathbf{R}\text{-a.s.}$$

*if and only if*

$$\mathbf{R}[X_2 \in \cdot | \mathcal{G}_1 \vee \mathcal{G}_3] \sim \mathbf{R}[X_2 \in \cdot | \mathcal{G}_3] \quad \mathbf{R}\text{-a.s.}$$

**PROOF.** This follows from [40, Lemma 3.6] and the existence of a measurable version of the Radon-Nikodym density between kernels [10, V.58].  $\square$

The second is essentially the tower property of conditional expectations.

**LEMMA 3.7.** *Let  $X_1, X_2, X_3$  be random variables taking values in Polish spaces  $H_1, H_2, H_3$ , respectively, and define the  $\sigma$ -fields  $\mathcal{G}_i = \sigma\{X_i\}$ . Moreover, let  $K : H_2 \times \mathcal{B}(H_1) \rightarrow [0, 1]$  be a transition kernel, and suppose that*

$$\mathbf{P}[X_1 \in \cdot | \mathcal{G}_2 \vee \mathcal{G}_3] \sim K(X_2, \cdot) \quad \mathbf{P}\text{-a.s.}$$

*Then we also have*

$$\mathbf{P}[X_1 \in \cdot | \mathcal{G}_2] \sim K(X_2, \cdot) \quad \mathbf{P}\text{-a.s.}$$



PROOF. Using existence of a measurable version of the Radon-Nikodym density between kernels [10, V.58], the assumption implies that there is a measurable function  $h : H_1 \times H_2 \times H_3 \rightarrow ]0, \infty[$  such that

$$\mathbf{P}[X_1 \in A | \mathcal{G}_2 \vee \mathcal{G}_3] = \int \mathbf{1}_A(x) h(x, X_2, X_3) K(X_2, dx) \quad \mathbf{P}\text{-a.s.}$$

for every  $A \in \mathcal{B}(H_1)$ . By the tower property

$$\mathbf{P}[X_1 \in A | \mathcal{G}_2] = \int \mathbf{1}_A(x) \int h(x, X_2, x') \mathbf{P}_{X_2}(dx') K(X_2, dx) \quad \mathbf{P}\text{-a.s.},$$

where we fix a version of the conditional probability  $\mathbf{P}_{X_2} = \mathbf{P}[X_3 \in \cdot | \mathcal{G}_2]$ . As  $\mathcal{B}(H_1)$  is countably generated, the  $\mathbf{P}$ -exceptional set can be chosen independent of  $A$  by a monotone class argument. This yields the claim.  $\square$

The third lemma is a total variation bound for conditional probabilities.

LEMMA 3.8. *Let  $H_1, H_2, H_3$  be Polish spaces, and let  $\mathbf{R}$  be a probability on  $H_1 \times H_2 \times H_3$ . Define  $X_i(x_1, x_2, x_3) = x_i$  and  $\mathcal{G}_i = \sigma\{X_i\}$ . Then*

$$\begin{aligned} \mathbf{E}_{\mathbf{R}}[|\mathbf{R}[X_1 \in \cdot | \mathcal{G}_2 \vee \mathcal{G}_3] - \mathbf{R}[X_1 \in \cdot | \mathcal{G}_3]|] \\ \leq 2 \mathbf{E}_{\mathbf{R}}[|\mathbf{R}[X_1, X_3 \in \cdot | \mathcal{G}_2] - \mathbf{R}[X_1, X_3 \in \cdot]|]. \end{aligned}$$

PROOF. We fix versions of the regular conditional probabilities  $\mathbf{R}_{X_2, X_3} = \mathbf{R}[X_1 \in \cdot | \mathcal{G}_2 \vee \mathcal{G}_3]$ ,  $\mathbf{R}_{X_3} = \mathbf{R}[X_1 \in \cdot | \mathcal{G}_3]$ , and  $\mathbf{R}^{X_2} = \mathbf{R}[X_1, X_3 \in \cdot | \mathcal{G}_2]$ , and let  $\tilde{\mathbf{R}}^{x_2} = \mathbf{R}^{x_2}[X_3 \in \cdot]$ . First, we claim that for  $\mathbf{R}$ -a.e.  $x_2$

$$\mathbf{R}^{x_2}(A) = \int \mathbf{1}_A(x_1, x_3) \mathbf{R}_{x_2, x_3}(dx_1) \tilde{\mathbf{R}}^{x_2}(dx_3) \quad \text{for all } A \in \mathcal{G}_1 \vee \mathcal{G}_3.$$

Indeed, as  $\mathbf{E}[A | \mathcal{G}_2] = \mathbf{E}[\mathbf{E}[A | \mathcal{G}_2 \vee \mathcal{G}_3] | \mathcal{G}_2]$ , the statement holds a.s. for all  $A$  in a countable generating algebra for  $\mathcal{G}_1 \vee \mathcal{G}_3$ , and the claim follows by the monotone class theorem. Now let  $\mathbf{R}_0^{x_2}(dx_1, dx_3) = \mathbf{R}_{x_3}(dx_1) \tilde{\mathbf{R}}^{x_2}(dx_3)$ . Then

$$\int \|\mathbf{R}_{x_2, x_3} - \mathbf{R}_{x_3}\| \tilde{\mathbf{R}}^{x_2}(dx_3) = \|\mathbf{R}^{x_2} - \mathbf{R}_0^{x_2}\| \leq \|\mathbf{R}^{x_2} - \mathbf{R}\|_{\mathcal{G}_1 \vee \mathcal{G}_3} + \|\tilde{\mathbf{R}}^{x_2} - \mathbf{R}\|_{\mathcal{G}_3}.$$

The proof is easily completed by integrating both sides.  $\square$

We now proceed to the proof of Theorem 3.5.

PROOF OF THEOREM 3.5. The nondegeneracy assumption states

$$\mathbf{P}[Y_{1,n-1} \in \cdot | \mathcal{F}_{-\infty,0} \vee \mathcal{F}_{n,\infty}] \sim \mathbf{P}[Y_{1,n-1} \in \cdot | \mathcal{F}_{-\infty,0}^Y \vee \mathcal{F}_{n,\infty}^Y] \quad \mathbf{P}\text{-a.s.}$$

Therefore, by Lemma 3.7, we obtain

$$\mathbf{P}[Y_{1,n-1} \in \cdot | \mathcal{F}_{-\infty,0} \vee \mathcal{F}_{n,\infty}^Y] \sim \mathbf{P}[Y_{1,n-1} \in \cdot | \mathcal{F}_{-\infty,0}^Y \vee \mathcal{F}_{n,\infty}^Y] \quad \mathbf{P}\text{-a.s.}$$

It follows that

$$\mathbf{P}[Y_{1,n-1} \in \cdot | \mathcal{F}_{-\infty,0} \vee \mathcal{F}_{n,\infty}] \sim \mathbf{P}[Y_{1,n-1} \in \cdot | \mathcal{F}_{-\infty,0}^Y \vee \mathcal{F}_{n,\infty}^Y] \quad \mathbf{P}\text{-a.s.},$$

which yields using Lemma 3.6

$$\mathbf{P}[Z_{n,\infty} \in \cdot | \mathcal{F}^Y \vee \mathcal{F}_{-\infty,0}^Z] \sim \mathbf{P}[Z_{n,\infty} \in \cdot | \mathcal{F}_{-\infty,0}^Y \vee \mathcal{F}_{n,\infty}^Y] \quad \mathbf{P}\text{-a.s.}$$

Therefore, if we choose any version  $\mathbf{P}_{Y,n}^{Z-} = \mathbf{P}[Z \in \cdot | \mathcal{F}_{-\infty,0}^Y \vee \mathcal{F}_{n,\infty}^Y \vee \mathcal{F}_{-\infty,0}^Z]$ ,

$$\mathbf{P}_y^{z-} |_{\mathcal{F}_{n,\infty}^Z} \sim \mathbf{P}_{y,n}^{z-} |_{\mathcal{F}_{n,\infty}^Z} \quad \forall n \geq 1 \quad \text{for } \mathbf{P}\text{-a.e. } (z, y)$$

(note that we define  $\mathbf{P}_{y,n}^{z-}$  as a function of the entire path  $y$  for simplicity of notation; by construction,  $\mathbf{P}_{y,n}^{z-}$  depends on  $y_{-\infty,0}, y_{n,\infty}, z_-$  only). By condition 3 of Theorem 3.1, to complete the proof it suffices to show that

$$\inf_n \|\mathbf{P}_{y,n}^{z-} - \mathbf{P}_{y,n}^{z'}\|_{\mathcal{F}_{n,\infty}^Z} < 2 \quad \text{for } \mathbf{Q}\text{-a.e. } (z, z', y),$$

where we define  $\mathbf{Q} = (\mathbf{P}_y \otimes \mathbf{P}_y)\mathbf{Y}$  as in section 3.1.

Fix versions of the regular conditional probabilities  $\mathbf{P}_{Y,n} = \mathbf{P}[Z \in \cdot | \mathcal{F}_{n,\infty}^Y]$  and  $\mathbf{P}^{X-} = \mathbf{P}[\cdot | \mathcal{F}_{-\infty,0}]$  (where  $X = (Z, Y)$ ). By Lemma 3.8, we have

$$\mathbf{E}\mathbf{Q} \left[ \|\mathbf{P}_{Y,n}^{Z-} - \mathbf{P}_{Y,n}^{Z'}\|_{\mathcal{F}_{n,\infty}^Z} \right] \leq 2 \mathbf{E} \left[ \|\mathbf{P}_{Y,n}^{Z-} - \mathbf{P}_{Y,n}\|_{\mathcal{F}_{n,\infty}^Z} \right] \leq 4 \mathbf{E} [\|\mathbf{P}^{X-} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}}].$$

As  $X = (Z, Y)$  is absolutely regular, Corollary 2.8 gives

$$\mathbf{E}\mathbf{Q} \left[ \inf_n \|\mathbf{P}_{Y,n}^{Z-} - \mathbf{P}_{Y,n}^{Z'}\|_{\mathcal{F}_{n,\infty}^Z} \right] \leq 4 \inf_n \mathbf{E} [\|\mathbf{P}^{X-} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}}] = 0.$$

Thus the requisite property is established.  $\square$

3.3. *One-sided observations.* Theorem 3.1 establishes conditional ergodicity of  $Z$  given the entire observation  $\sigma$ -field  $\mathcal{F}^Y$ . This allows us to control the behavior of the conditional distributions  $\mathbf{P}[Z_{n,\infty} \in \cdot | \mathcal{F}^Y]$  as  $n \rightarrow \infty$ . In contrast, the ergodic theory of nonlinear filters (section 4) is concerned with the “causal” setting where one considers the conditional distributions  $\mathbf{P}[X_{n,\infty} \in \cdot | \mathcal{F}_{0,n}^Y]$  as  $n \rightarrow \infty$ . The latter requires a one-sided version of our results where we only condition on  $\mathcal{F}_+^Y = \mathcal{F}_{0,\infty}^Y$ . Unfortunately, two-sided conditioning was essential to obtain a conditional zero-two law: if we had replaced  $\mathcal{F}^Y$  by  $\mathcal{F}_+^Y$  in section 3.1, for example, then the coupled measure  $\mathbf{Q}$  would be nonstationary and the key Lemmas 3.2 and 3.3 would fail.

We must therefore develop an additional technique to deduce one-sided conditional ergodicity results from their two-sided counterparts. To this end we prove the following result, which will suffice for our purposes.

PROPOSITION 3.9 (One-sided conditioning). *Suppose that the stationary process  $(Z_k, Y_k)_{k \in \mathbb{Z}}$  is absolutely regular and nondegenerate. Then*

$$\mathbf{P}[Z_{0,\infty} \in \cdot | \mathcal{F}_+^Y] \sim \mathbf{P}[Z_{0,\infty} \in \cdot | \mathcal{F}^Y] \quad \mathbf{P}\text{-a.s.}$$

Before we prove this result, let us use it to establish the key  $\sigma$ -field identity that will be needed in the ergodic theory of nonlinear filters (section 4).

COROLLARY 3.10. *Suppose that the stationary process  $(Z_k, Y_k)_{k \in \mathbb{Z}}$  is absolutely regular and nondegenerate. Then the following holds:*

$$\bigcap_{n \geq 0} \mathcal{F}_+^Y \vee \mathcal{F}_{n,\infty}^Z = \mathcal{F}_+^Y \quad \text{mod } \mathbf{P}.$$

PROOF. Using a monotone class argument as in the proof of Lemma 3.8,

$$\int \mathbf{P}_y^{z-}(A) \mathbf{P}_y(dz) = \mathbf{P}_y(A) \quad \text{for all } A \in \mathcal{F}^Z$$

holds for  $\mathbf{Y}$ -a.e.  $y$ . Therefore, by Theorems 3.5 and 3.1, we find that

$$\|\mathbf{P}_y^{z-} - \mathbf{P}_y\|_{\mathcal{F}_{n,\infty}^Z} \leq \int \|\mathbf{P}_y^{z-} - \mathbf{P}_y^{z'}\|_{\mathcal{F}_{n,\infty}^Z} \mathbf{P}_y(dz') \xrightarrow{n \rightarrow \infty} 0$$

for  $\mathbf{P}$ -a.e.  $(z, y)$ . As  $\|\mathbf{P}_y^{z-} - \mathbf{P}_y\|_{\mathcal{F}_{n,\infty}^Z} \rightarrow \|\mathbf{P}_y^{z-} - \mathbf{P}_y\|_{\mathcal{A}^Z}$  as  $n \rightarrow \infty$ , applying again Theorems 3.5 and 3.1 shows that  $\mathcal{A}^Z$  is  $\mathbf{P}_y$ -trivial for  $\mathbf{Y}$ -a.e.  $y$ .

Fix a version of the regular conditional probability  $\mathbf{P}_{Y_+} = \mathbf{P}[\cdot | \mathcal{F}_+^Y]$ , where  $Y_+ = (Y_k)_{k \geq 0}$ . By Proposition 3.9,  $\mathbf{P}_y$  and  $\mathbf{P}_{y_+}$  are equivalent on  $\mathcal{F}_{0,\infty}^Z$  for  $\mathbf{Y}$ -a.e.  $y$ . It follows that  $\mathcal{A}^Z$  is also  $\mathbf{P}_{y_+}$ -trivial for  $\mathbf{Y}$ -a.e.  $y$ . In particular,

$$\mathbf{P}_{y_+}[A | \mathcal{F}_{n,\infty}^Z] \xrightarrow{n \rightarrow \infty} \mathbf{P}_{y_+}[A] \quad \mathbf{P}_{y_+}\text{-a.s.} \quad \text{for } \mathbf{Y}\text{-a.e. } y$$

holds for any  $A \in \mathcal{F}$ . By Lemma 3.11 below, this implies that

$$\mathbf{P}[A|\mathcal{F}_+^Y \vee \mathcal{F}_{n,\infty}^Z] \xrightarrow{n \rightarrow \infty} \mathbf{P}[A|\mathcal{F}_+^Y] \quad \mathbf{P}\text{-a.s.} \quad \text{for every } A \in \mathcal{F}.$$

This evidently establishes the claim.  $\square$

We used above the following basic fact on repeated conditioning [46, p. 95–96]. We omit the proof, which is essentially along the lines of Lemma 3.2.

LEMMA 3.11. *Let  $H_1, H_2, H_3$  be Polish spaces, let  $\mathbf{R}$  be a probability on  $H_1 \times H_2 \times H_3$  and  $X_i(x_1, x_2, x_3) = x_i$ ,  $\mathcal{G}_i = \sigma\{X_i\}$ . Choose any versions of the regular conditional probabilities  $\mathbf{R}_{X_1} = \mathbf{R}[X_2, X_3 \in \cdot | \mathcal{G}_1]$  and  $\mathbf{R}_{X_1, X_2} = \mathbf{R}[X_3 \in \cdot | \mathcal{G}_1 \vee \mathcal{G}_2]$ , and let  $\mathbf{R}^1 = \mathbf{R}[X_1 \in \cdot]$ . Then for  $\mathbf{R}^1$ -a.e.  $x_1 \in H_1$ ,*

$$\mathbf{R}_{x_1, X_2}(A) = \mathbf{R}_{x_1}[X_3 \in A | \mathcal{G}_2] \quad \mathbf{R}_{x_1}\text{-a.s.} \quad \text{for all } A \in \mathcal{G}_3.$$

We now turn to the proof of Proposition 3.9. The essential difficulty is that we must show equivalence of two measures on an infinite time interval. The following lemma provides a simple tool for this purpose.

LEMMA 3.12. *Let  $H$  be a Polish space, and let  $\mu, \nu$  be probability measures on  $H^{\mathbb{N}}$ . Denote by  $X_i : H^{\mathbb{N}} \rightarrow H$  the coordinate projections  $X_i(x) = x_i$ , and define the  $\sigma$ -fields  $\mathcal{G}_{m,n} = \sigma\{X_{m,n}\}$  for  $m \leq n$ . If we have*

$$\mu[X_{1,n-1} \in \cdot | \mathcal{G}_{n,\infty}] \ll \nu[X_{1,n-1} \in \cdot | \mathcal{G}_{n,\infty}] \quad \mu|_{\mathcal{G}_{n,\infty}} \wedge \nu|_{\mathcal{G}_{n,\infty}}\text{-a.s.}$$

for all  $n < \infty$ , and if in addition

$$\|\mu - \nu\|_{\mathcal{G}_{n,\infty}} \xrightarrow{n \rightarrow \infty} 0,$$

then  $\mu \ll \nu$  on  $\mathcal{G}_{1,\infty}$ .

PROOF. Let  $\mu_n = \mu|_{\mathcal{G}_{n,\infty}}$  and  $\nu_n = \nu|_{\mathcal{G}_{n,\infty}}$ . Choose any  $A \in \mathcal{G}_{1,\infty}$  such that  $\nu(A) = 0$ . Then  $\nu[A|\mathcal{G}_{n,\infty}] = 0$   $\nu_n$ -a.s., and therefore  $\mu[A|\mathcal{G}_{n,\infty}] = 0$   $\mu_n \wedge \nu_n$ -a.s. by the first assumption. But using the second assumption

$$\mu(A) = \mathbf{E}_{\mu_n \wedge \nu_n}[\mu[A|\mathcal{G}_{n,\infty}]] + \mathbf{E}_{\mu_n - \mu_n \wedge \nu_n}[\mu[A|\mathcal{G}_{n,\infty}]] \leq 1 - \|\mu_n \wedge \nu_n\| \xrightarrow{n \rightarrow \infty} 0,$$

where we used  $\|\mu - \nu\|_{\mathcal{G}_{n,\infty}} = 2(1 - \|\mu_n \wedge \nu_n\|)$ . Thus  $\mu \ll \nu$  on  $\mathcal{G}_{1,\infty}$ .  $\square$

We can now complete the proof of Proposition 3.9.

PROOF OF PROPOSITION 3.9. By Lemma 3.6, it suffices to show that

$$\mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_+^Y] \sim \mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_+] \quad \mathbf{P}\text{-a.s.}$$

Fix versions of the regular conditional probabilities

$$\begin{aligned} \mathbf{P}_{Y_{0,\infty}} &= \mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_+^Y], \\ \mathbf{P}_{X_{0,\infty}} &= \mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_+], \\ \mathbf{P}_{Y_{0,\infty}, Y_{-\infty,-n}} &= \mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_+^Y, \mathcal{F}_{-\infty,-n}^Y], \\ \mathbf{P}_{X_{0,\infty}, Y_{-\infty,-n}} &= \mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_+, \mathcal{F}_{-\infty,-n}^Y]. \end{aligned}$$

We will show that  $\mathbf{P}_{Y_{0,\infty}} \sim \mathbf{P}_{X_{0,\infty}}$   $\mathbf{P}$ -a.s. by applying Lemma 3.12.

First, we claim that

$$\|\mathbf{P}_{y_{0,\infty}} - \mathbf{P}_{x_{0,\infty}}\|_{\mathcal{F}_{-\infty,-m}^Y} \xrightarrow{m \rightarrow \infty} 0 \quad \text{for } \mathbf{P}\text{-a.e. } x = (z, y).$$

Indeed, note that by the triangle inequality and Jensen's inequality

$$\begin{aligned} &\|\mathbf{P}[Y_{-\infty,-m} \in \cdot | \mathcal{F}_+^Y] - \mathbf{P}[Y_{-\infty,-m} \in \cdot | \mathcal{F}_+]\| \\ &\leq \|\mathbf{P}[\cdot | \mathcal{F}_+] - \mathbf{P}\|_{\mathcal{F}_{-\infty,-m}} + \mathbf{E}[\|\mathbf{P}[\cdot | \mathcal{F}_+] - \mathbf{P}\|_{\mathcal{F}_{-\infty,-m}} | \mathcal{F}_+^Y]. \end{aligned}$$

By Corollary 2.8, it suffices to show that the time-reversed process  $(X_{-k})_{k \in \mathbb{Z}}$  is absolutely regular. But it is clear from Definition 2.7 that the absolute regularity property of a stationary sequence is invariant under time reversal. As we assumed absolute regularity of  $(X_k)_{k \in \mathbb{Z}}$ , the claim follows.

Next, we claim that for  $\mathbf{P}$ -a.e.  $x = (z, y)$

$$\mathbf{P}_{y_{0,\infty}, Y_{-\infty,-m}} \sim \mathbf{P}_{x_{0,\infty}, Y_{-\infty,-m}} \quad \mathbf{P}_{y_{0,\infty}}|_{\mathcal{F}_{-\infty,-m}^Y} \wedge \mathbf{P}_{x_{0,\infty}}|_{\mathcal{F}_{-\infty,-m}^Y} \text{-a.s.}$$

Indeed, as in the proof of Theorem 3.5, the nondegeneracy assumption yields

$$\mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_{-\infty,-m}^Y \vee \mathcal{F}_+^Y] \sim \mathbf{P}[Y_{-\infty,-1} \in \cdot | \mathcal{F}_{-\infty,-m}^Y \vee \mathcal{F}_+] \quad \mathbf{P}\text{-a.s.}$$

Thus there is a set  $H \subseteq (G \times F)^{\mathbb{N}} \times F^{\mathbb{N}}$  with  $\mathbf{P}[(X_{0,\infty}, Y_{-\infty,-m}) \in H] = 1$  such that  $\mathbf{P}_{y_{0,\infty}, y_{-\infty,-m}} \sim \mathbf{P}_{x_{0,\infty}, y_{-\infty,-m}}$  for all  $(x_{0,\infty}, y_{-\infty,-m}) \in H$ , and

$$\int \mathbf{1}_H(x_{0,\infty}, y_{-\infty,-m}) \mathbf{P}_{x_{0,\infty}}(dy) = 1 \quad \text{for } \mathbf{P}\text{-a.e. } x.$$

It follows that  $\mathbf{P}_{y_{0,\infty}, Y_{-\infty,-m}} \sim \mathbf{P}_{x_{0,\infty}, Y_{-\infty,-m}}$  holds  $\mathbf{P}_{x_{0,\infty}}|_{\mathcal{F}_{-\infty,-m}^Y}$ -a.s., and thus *a fortiori*  $\mathbf{P}_{y_{0,\infty}}|_{\mathcal{F}_{-\infty,-m}^Y} \wedge \mathbf{P}_{x_{0,\infty}}|_{\mathcal{F}_{-\infty,-m}^Y}$ -a.s., for  $\mathbf{P}$ -a.e.  $x$ .

Finally, using Lemma 3.11 we conclude that for  $\mathbf{P}$ -a.e.  $x = (z, y)$ , we have verified the assumptions of Lemma 3.12 for the measures  $\mathbf{P}_{y_{0,\infty}}$  and  $\mathbf{P}_{x_{0,\infty}}$ . Thus we have shown  $\mathbf{P}_{Y_{0,\infty}} \sim \mathbf{P}_{X_{0,\infty}}$   $\mathbf{P}$ -a.s., and the proof is complete.  $\square$

**4. Ergodicity of the filter.** Let  $(X_n, Y_n)_{n \geq 0}$  be a Markov chain. We interpret  $(X_n)_{n \geq 0}$  as the unobserved component of the model, while  $(Y_n)_{n \geq 0}$  is the observable process. In this setting, there are two distinct levels on which the conditional ergodic theory of the model can be investigated.

In the previous section, we investigated directly the ergodic properties of the unobserved process  $(X_n)_{n \geq 0}$  conditionally on the observations  $(Y_n)_{n \geq 0}$ . This setting is of interest if the entire observation sequence  $(Y_n)_{n \geq 0}$  is available *a priori*. In contrast, it is often of interest to consider the setting of *causal* conditioning, where we wish to infer the current state  $X_n$  of the unobserved process given only the history of observations to date  $\mathcal{F}_{0,n}^Y$ . An object of central importance in this setting is the *nonlinear filter*

$$\pi_n = \mathbf{P}[X_n \in \cdot | \mathcal{F}_{0,n}^Y].$$

Evidently, the filtering process  $(\pi_n)_{n \geq 0}$  is a measure-valued process that is adapted to the observation filtration  $\mathcal{F}_{0,n}^Y$ . The goal of this section is to investigate the stability and ergodic properties of the filter  $(\pi_n)_{n \geq 0}$ .

In section 4.1 we will develop the basic setting and notation to be used throughout this section. In section 4.2 we develop a local stability result for the nonlinear filter, which is in essence the filtering counterpart of the local zero-two laws of section 2.1. In section 4.3 we apply the local stability result to develop a number of general stability and ergodicity results for the nonlinear filter that are applicable to infinite-dimensional models. Finally, in section 4.4 we will extend our results to the continuous time setting.

The filter stability and ergodicity results developed in this section provided the main motivation for the theory developed in this paper; their broad applicability will be illustrated in section 5 below.

#### 4.1. Setting and notation.

**4.1.1. The canonical setup.** Throughout this section, we will consider the bivariate stochastic process  $(X_n, Y_n)_{n \in \mathbb{Z}}$ , where  $X_n$  takes values in the Polish space  $E$  and  $Y_n$  takes values in the Polish space  $F$ . We realize this process on the canonical path space  $\Omega = \Omega^X \times \Omega^Y$  with  $\Omega^X = E^{\mathbb{Z}}$  and  $\Omega^Y = F^{\mathbb{Z}}$ , such that  $X_n(x, y) = x(n)$  and  $Y_n(x, y) = y(n)$ . Denote by  $\mathcal{F}$  the Borel  $\sigma$ -field of  $\Omega$ , and define  $X_{m,n} = (X_k)_{m \leq k \leq n}$ ,  $Y_{m,n} = (Y_k)_{m \leq k \leq n}$ , and

$$\mathcal{F}_{m,n}^X = \sigma\{X_{m,n}\}, \quad \mathcal{F}_{m,n}^Y = \sigma\{Y_{m,n}\}, \quad \mathcal{F}_{m,n} = \mathcal{F}_{m,n}^X \vee \mathcal{F}_{m,n}^Y$$

for  $m \leq n$ . For simplicity of notation, we define the  $\sigma$ -fields

$$\mathcal{F}^X = \mathcal{F}_{-\infty, \infty}^X, \quad \mathcal{F}^Y = \mathcal{F}_{-\infty, \infty}^Y, \quad \mathcal{F}_+^X = \mathcal{F}_{0, \infty}^X, \quad \mathcal{F}_+^Y = \mathcal{F}_{0, \infty}^Y, \quad \mathcal{F}_+ = \mathcal{F}_{0, \infty}.$$

Finally, we denote by  $Y$  the  $F^{\mathbb{Z}}$ -valued random variable  $(Y_k)_{k \in \mathbb{Z}}$ , and the canonical shift  $\Theta : \Omega \rightarrow \Omega$  is defined as  $\Theta(x, y)(m) = (x(m+1), y(m+1))$ .

For any Polish space  $Z$ , we denote by  $\mathcal{B}(Z)$  its Borel  $\sigma$ -field, and by  $\mathcal{P}(Z)$  the space of all probability measures on  $Z$  endowed with the weak convergence topology (thus  $\mathcal{P}(Z)$  is again Polish). Let us recall that any probability kernel  $\rho : Z \times \mathcal{B}(Z') \rightarrow [0, 1]$  may be equivalently viewed as a  $\mathcal{P}(Z')$ -valued random variable  $z \mapsto \rho(z, \cdot)$  on  $(Z, \mathcal{B}(Z))$ . For notational convenience, we will implicitly identify probability kernels and random probability measures in the sequel. The notation for total variation distance is as in section 2.

**4.1.2. The Markov model.** The basic model of this section is defined by a Markov transition kernel  $P : E \times F \times \mathcal{B}(E \times F) \rightarrow [0, 1]$  on  $E \times F$ . Denote by  $\mathbf{P}^\mu$  the probability measure on  $\mathcal{F}_+$  such that  $(X_n, Y_n)_{n \geq 0}$  is a Markov chain with transition kernel  $P$  and initial law  $(X_0, Y_0) \sim \mu \in \mathcal{P}(E \times F)$ . For any  $(x, y) \in E \times F$ , we will denote for simplicity the law of the Markov chain started at the point mass  $(X_0, Y_0) = (x, y)$  as  $\mathbf{P}^{x, y} = \mathbf{P}^{\delta_x \otimes \delta_y}$ .

We now impose the following standing assumption.

**STANDING ASSUMPTION.** The Markov transition kernel  $P$  admits an invariant probability measure  $\lambda \in \mathcal{P}(E \times F)$ , that is,  $\lambda P = \lambda$ .

Let us emphasize that we do not rule out at this point the existence of more than one invariant probability; we simply fix one invariant probability  $\lambda$  in what follows. Our results will be stated in terms of  $\lambda$ .

Note that by construction,  $(X_n, Y_n)_{n \geq 0}$  is a stationary Markov chain under  $\mathbf{P}^\lambda$ . We can therefore naturally extend  $\mathbf{P}^\lambda$  to  $\mathcal{F}$  such that the two-sided process  $(X_n, Y_n)_{n \in \mathbb{Z}}$  is the stationary Markov chain with invariant probability  $\lambda$  under  $\mathbf{P}^\lambda$ . For simplicity, we will frequently write  $\mathbf{P} = \mathbf{P}^\lambda$ .

**4.1.3. The nonlinear filter.** The Markov chain  $(X_n, Y_n)_{n \geq 0}$  consists of two components:  $(X_n)_{n \geq 0}$  represents the unobservable component of the model, while  $(Y_n)_{n \geq 0}$  represents the observable component. As  $(X_n)_{n \geq 0}$  is presumed to be observable, we are interested at time  $n$  in the conditional distribution given the observation history to date  $Y_0, \dots, Y_n$ . To this end, we will introduce for every  $\mu \in \mathcal{P}(E \times F)$  the following random measures:

$$\Pi_n^\mu = \mathbf{P}^\mu[\cdot | \mathcal{F}_{0,n}^Y], \quad \pi_n^\mu = \mathbf{P}^\mu[X_n \in \cdot | \mathcal{F}_{0,n}^Y].$$

The  $\mathcal{P}(E)$ -valued process  $(\pi_n^\mu)_{n \geq 0}$  is called the *nonlinear filter* started at  $\mu$ . This is ultimately our main object of interest. However, we will find it convenient to investigate the full conditional distributions  $\Pi_n^\mu$ . When  $\mu = \lambda$  is the invariant measure, we will write  $\Pi_n = \Pi_n^\lambda$  and  $\pi_n = \pi_n^\lambda$ .

REMARK 4.1. Note that  $\Pi_n^\mu, \pi_n^\mu$  are  $\mathcal{F}_{0,n}^Y$ -measurable kernels. That is,  $\pi_n^\mu : \Omega \times \mathcal{B}(E) \rightarrow [0, 1]$  can be written as  $\pi_n^\mu(A) = \pi_n^\mu[Y_{0,n}; A]$  for  $A \in \mathcal{B}(E)$ . We will mostly suppress the dependence on  $Y_{0,n}$  for notational convenience.

4.2. *A local stability result.* The main tool that we will develop to investigate the ergodic theory of nonlinear filters is a *local* stability result for the conditional distributions  $\Pi_n^\mu$ . To this end, we fix in this subsection a countably generated local  $\sigma$ -field  $\mathcal{E}^0 \subseteq \mathcal{B}(E)$  as in section 2.1, and define

$$\mathcal{F}_{m,n}^0 = \mathcal{F}_{m,n}^Y \vee \bigvee_{m \leq k \leq n} X_k^{-1}(\mathcal{E}^0), \quad m < n.$$

Let us emphasize that the localization pertains only to the unobserved component  $X_k$ : it is essential for our results that the entire observation variable  $Y_k$  is included in the local filtration  $\mathcal{F}_{m,n}^0$ . In practice, this typically implies that the unobserved process may be infinite-dimensional, but the observed process must be finite-dimensional (see section 5 for examples).

As in section 3, we require two basic assumptions. The first assumption states that the model  $(X_n, Y_n)_{n \geq 0}$  is locally ergodic. The second assumption provides a notion of nondegeneracy that is adapted to the present setting.

ASSUMPTION 4.2 (Local ergodicity). The following holds:

$$\|\mathbf{P}^{x,y} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \lambda\text{-a.e. } (x, y) \in E \times F.$$

ASSUMPTION 4.3 (Nondegeneracy). There exist Markov transition kernels  $P_0 : E \times \mathcal{B}(E) \rightarrow [0, 1]$  and  $Q : F \times \mathcal{B}(F) \rightarrow [0, 1]$  such that

$$P(x, y, dx', dy') = g(x, y, x', y') P_0(x, dx') Q(y, dy')$$

for some strictly positive measurable function  $g : E \times F \times E \times F \rightarrow ]0, \infty[$ .

Note that Assumption 4.2 is characterized by Theorem 2.2, which yields a general tool to verify this assumption. Assumption 4.3 is easily verified in practice as it is stated directly in terms of the underlying model.

The main result of this section is as follows.

THEOREM 4.4 (Local filter stability). *Suppose that Assumptions 4.2 and 4.3 hold. Then for any initial probability  $\mu \in \mathcal{P}(E \times F)$  such that*

$$\mu(E \times \cdot) \ll \lambda(E \times \cdot) \quad \text{and} \quad \|\Pi_0^\mu - \Pi_0\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.},$$

*we have*

$$\|\Pi_n^\mu - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.} \quad \text{for any } r \in \mathbb{N}.$$

*If  $\mu(E \times \cdot) \sim \lambda(E \times \cdot)$ , the convergence holds also  $\mathbf{P}$ -a.s.*



REMARK 4.5. When interpreting this result, we must take care to ensure that the relevant quantities are well defined. Recall that  $\Pi_n^\mu$ , as a regular conditional probability, is defined uniquely up to a  $\mathbf{P}^\mu|_{\mathcal{F}_{0,n}^Y}$ -null set only. As part of the proof, we will in fact show that under the stated assumptions  $\mathbf{P}^\mu|_{\mathcal{F}_+^Y} \ll \mathbf{P}|_{\mathcal{F}_+^Y}$  (resp.  $\mathbf{P}^\mu|_{\mathcal{F}_+^Y} \sim \mathbf{P}|_{\mathcal{F}_+^Y}$  when  $\mu(E \times \cdot) \sim \lambda(E \times \cdot)$ ). This ensures that  $\Pi_n$  is  $\mathbf{P}^\mu$ -a.s. uniquely defined (resp.  $\Pi_n^\mu$  is  $\mathbf{P}$ -a.s. uniquely defined).

If we strengthen Assumption 4.2 as in Theorem 2.1, we obtain:

COROLLARY 4.6. *Suppose that Assumption 4.3 holds and that*

$$\|\mathbf{P}^{x,y} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } (x,y) \in E \times F.$$

*Then for any  $\mu \in \mathcal{P}(E \times F)$  such that  $\mu(E \times \cdot) \ll \lambda(E \times \cdot)$ , we have*

$$\|\Pi_n^\mu - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.} \quad \text{for any } r \in \mathbb{N}.$$

*If  $\mu(E \times \cdot) \sim \lambda(E \times \cdot)$ , the convergence holds also  $\mathbf{P}$ -a.s.*

PROOF. The assumption clearly implies Assumption 4.2. Moreover,

$$\begin{aligned} \|\Pi_0^\mu - \Pi_0\|_{\mathcal{F}_{n,\infty}^0} &\leq \|\mathbf{P}^\mu[\cdot|Y_0] - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} + \|\mathbf{P}[\cdot|Y_0] - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} \\ &\leq \mathbf{E}^\mu[\|\mathbf{P}^{X_0,Y_0} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} | Y_0] + \mathbf{E}[\|\mathbf{P}^{X_0,Y_0} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} | Y_0]. \end{aligned}$$

Thus  $\|\Pi_0^\mu - \Pi_0\|_{\mathcal{F}_{n,\infty}^0} \rightarrow 0$   $\mathbf{P}^\mu$ -a.s. It remains to apply Theorem 4.4.  $\square$

We now turn to the proof of Theorem 4.4. We begin with a trivial consequence of the Bayes formula that we formulate for completeness.

LEMMA 4.7. *Let  $\mu, \nu$  be probability measures on a Polish space  $H$ , and let  $\mathcal{G} \subseteq \mathcal{B}(H)$  be a  $\sigma$ -field. If  $\mu \sim \nu$ , then  $\mu[\cdot|\mathcal{G}] \sim \nu[\cdot|\mathcal{G}]$   $\mu$ -a.s. and  $\nu$ -a.s.*

PROOF. Let  $\Lambda = d\mu/d\nu$ . Then for any  $A \in \mathcal{B}(H)$ , we have

$$\mu[A|\mathcal{G}] = \mathbf{E}_\nu \left[ \mathbf{1}_A \frac{\Lambda}{\mathbf{E}_\nu[\Lambda|\mathcal{G}]} \middle| \mathcal{G} \right] \quad \text{and} \quad \frac{\Lambda}{\mathbf{E}_\nu[\Lambda|\mathcal{G}]} > 0 \quad \mu\text{-a.s.}$$

by the Bayes formula and using  $\mu \sim \nu$ . As  $\mathcal{B}(H)$  is countably generated, the  $\mu$ -exceptional set can be chosen independent of  $A$  by a monotone class argument. Thus  $\mu[\cdot|\mathcal{G}] \sim \nu[\cdot|\mathcal{G}]$   $\mu$ -a.s., and also  $\nu$ -a.s. as  $\mu \sim \nu$ .  $\square$

To proceed, we first prove a key measure-theoretic identity that arises from the conditional ergodic theory developed in section 3 above.

LEMMA 4.8. *Suppose that Assumptions 4.2 and 4.3 hold. Then*

$$\bigcap_{n \geq 0} \mathcal{F}_+^Y \vee \mathcal{F}_{n,\infty}^0 = \mathcal{F}_+^Y \quad \text{mod } \mathbf{P}.$$

PROOF. Let  $(A_n)_{n \in \mathbb{N}}$  be a countable generating class for  $\mathcal{E}^0$ , and define  $Z_n = \iota(X_n)$  where  $\iota : E \rightarrow \{0, 1\}^{\mathbb{N}}$  is given by  $\iota(x) = (\mathbf{1}_{A_n}(x))_{n \in \mathbb{N}}$ . Then  $\mathcal{F}_{m,n}^0 = \mathcal{F}_{m,n}^Y \vee \mathcal{F}_{m,n}^Z$  by construction. It now suffices to show that the stationary process  $(Z_n, Y_n)_{n \in \mathbb{Z}}$  satisfies the assumptions of Corollary 3.10.

First, note that by Assumption 4.3 and the Markov property

$$\|\mathbf{P}[\cdot | \mathcal{F}_{-\infty,0}^0] - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} \leq \mathbf{E}[\|\mathbf{P}^{X_0, Y_0} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^0} | \mathcal{F}_{-\infty,0}^0] \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}\text{-a.s.}$$

Thus Corollary 2.8 yields absolute regularity of  $(Z_n, Y_n)_{n \in \mathbb{Z}}$ .

It remains to prove nondegeneracy (in the sense of Definition 3.4). To this end, we begin by noting that by the Markov property of  $(X_n, Y_n)_{n \in \mathbb{Z}}$

$$\mathbf{P}[Y_{1,n} \in \cdot | \mathcal{F}_{-\infty,0} \vee \mathcal{F}_{n+1,\infty}] = \mathbf{P}^{X_0, Y_0}[Y_{1,n} \in \cdot | X_{n+1}, Y_{n+1}].$$

Let  $\mathbf{R}^{x,y}$  be the probability on  $\mathcal{F}_+$  under which  $(X_k, Y_k)_{k \geq 0}$  is a Markov chain with transition kernel  $P_0 \otimes Q$  started as  $(X_0, Y_0) = (x, y)$ . Then  $\mathbf{P}^{x,y}|_{\mathcal{F}_{0,n+1}} \sim \mathbf{R}^{x,y}|_{\mathcal{F}_{0,n+1}}$  by Assumption 4.3. Therefore, by Lemma 4.7,

$$\mathbf{P}[Y_{1,n} \in \cdot | \mathcal{F}_{-\infty,0} \vee \mathcal{F}_{n+1,\infty}] \sim \mathbf{R}^{X_0, Y_0}[Y_{1,n} \in \cdot | X_{n+1}, Y_{n+1}] \quad \mathbf{P}\text{-a.s.}$$

But  $X_{0,n+1}$  and  $Y_{0,n+1}$  are independent under  $\mathbf{R}^{x,y}$ , so

$$\mathbf{R}^{X_0, Y_0}[Y_{1,n} \in \cdot | X_{n+1}, Y_{n+1}] = \mathbf{R}[Y_{1,n} \in \cdot | Y_0, Y_{n+1}] \quad \mathbf{P}\text{-a.s.}$$

where  $\mathbf{R} = \int \mathbf{R}^{x,y} \lambda(dx, dy)$ . Therefore, by Lemma 3.7,

$$\mathbf{P}[Y_{1,n} \in \cdot | \mathcal{F}_{-\infty,0}^0 \vee \mathcal{F}_{n+1,\infty}^0] \sim \mathbf{P}[Y_{1,n} \in \cdot | \mathcal{F}_{-\infty,0}^Y \vee \mathcal{F}_{n+1,\infty}^Y] \quad \mathbf{P}\text{-a.s.}$$

As this holds for any  $n \in \mathbb{N}$ , and using stationarity of  $\mathbf{P}$ , it follows readily that the process  $(Z_k, Y_k)_{k \in \mathbb{Z}}$  is nondegenerate (Definition 3.4).  $\square$

Armed with this result, we prove first a dominated stability lemma.

LEMMA 4.9. *Suppose that Assumptions 4.2 and 4.3 hold. Let  $\mathbf{Q}$  be a probability measure on  $\mathcal{F}_+$ , and define  $\Sigma_n = \mathbf{Q}[\cdot | \mathcal{F}_{0,n}^Y]$ . Suppose that*

$$\mathbf{Q}|_{\mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0} \ll \mathbf{P}|_{\mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0} \quad \text{for some } m \in \mathbb{N}.$$

*Then  $\|\Sigma_n - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \rightarrow 0$   $\mathbf{Q}$ -a.s. as  $n \rightarrow \infty$  for any  $r \in \mathbb{N}$ .*

PROOF. Fix any  $r$  and  $m$  as in the statement of the lemma, and let  $n \geq m + r$ . By the Bayes formula, we obtain for any set  $A \in \mathcal{F}_{n-r,\infty}^0$

$$\mathbf{Q}[A|\mathcal{F}_{0,n}^Y] = \frac{\mathbf{E}[\mathbf{1}_A \mathbf{E}[\Lambda|\mathcal{F}_+^Y \vee \mathcal{F}_{n-r,\infty}^0]|\mathcal{F}_{0,n}^Y]}{\mathbf{E}[\Lambda|\mathcal{F}_{0,n}^Y]} \quad \mathbf{Q}\text{-a.s.}, \quad \Lambda = \frac{d\mathbf{Q}|_{\mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0}}{d\mathbf{P}|_{\mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0}}$$

(we write  $\mathbf{E}[X] = \mathbf{E}_{\mathbf{P}}[X]$  for simplicity). We therefore have for  $A \in \mathcal{F}_{n-r,\infty}^0$

$$\Sigma_n(A) - \Pi_n(A) = \int \mathbf{1}_A \left\{ \frac{\mathbf{E}[\Lambda|\mathcal{F}_+^Y \vee \mathcal{F}_{n-r,\infty}^0]}{\mathbf{E}[\Lambda|\mathcal{F}_{0,n}^Y]} - 1 \right\} d\Pi_n \quad \mathbf{Q}\text{-a.s.}$$

As  $\mathcal{F}_{n-r,\infty}^0$  is countably generated, the  $\mathbf{Q}$ -exceptional set can be chosen independent of  $A$  using a monotone class argument. It follows that

$$\|\Sigma_n - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \leq \int \left| \frac{\mathbf{E}[\Lambda|\mathcal{F}_+^Y \vee \mathcal{F}_{n-r,\infty}^0]}{\mathbf{E}[\Lambda|\mathcal{F}_{0,n}^Y]} - 1 \right| d\Pi_n = \frac{\mathbf{E}[\Delta_n|\mathcal{F}_{0,n}^Y]}{\mathbf{E}[\Lambda|\mathcal{F}_{0,n}^Y]} \quad \mathbf{Q}\text{-a.s.},$$

where we have defined

$$\Delta_n = |\mathbf{E}[\Lambda|\mathcal{F}_+^Y \vee \mathcal{F}_{n-r,\infty}^0] - \mathbf{E}[\Lambda|\mathcal{F}_{0,n}^Y]|.$$

We now estimate

$$\mathbf{E}[\Delta_n|\mathcal{F}_{0,n}^Y] \leq \mathbf{E}[\Delta_n^u|\mathcal{F}_{0,n}^Y] + 2\mathbf{E}[\Lambda \mathbf{1}_{\Lambda > u}|\mathcal{F}_{0,n}^Y]$$

where

$$\Delta_n^u = |\mathbf{E}[\Lambda \mathbf{1}_{\Lambda \leq u}|\mathcal{F}_+^Y \vee \mathcal{F}_{n-r,\infty}^0] - \mathbf{E}[\Lambda \mathbf{1}_{\Lambda \leq u}|\mathcal{F}_{0,n}^Y]|.$$

By Lemma 4.8 and Hunt's lemma, we obtain  $\mathbf{E}[\Delta_n^u|\mathcal{F}_{0,n}^Y] \rightarrow 0$   $\mathbf{P}$ -a.s. as  $n \rightarrow \infty$ . Moreover, as  $\mathbf{E}[\Lambda|\mathcal{F}_+^Y] > 0$   $\mathbf{Q}$ -a.s., it follows that

$$\limsup_{n \rightarrow \infty} \|\Sigma_n - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \leq \frac{2\mathbf{E}[\Lambda \mathbf{1}_{\Lambda > u}|\mathcal{F}_+^Y]}{\mathbf{E}[\Lambda|\mathcal{F}_+^Y]} \quad \mathbf{Q}\text{-a.s.}$$

Letting  $u \rightarrow \infty$  completes the proof.  $\square$

The use the previous lemma, we will decompose  $\mathbf{P}^\mu$  on  $\mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0$  into an absolutely continuous component with respect to  $\mathbf{P}$  (to which Lemma 4.9 can be applied) and a remainder that is negligible as  $m \rightarrow \infty$ . The following lemma ensures that this can be done under the assumptions of Theorem 4.4.

LEMMA 4.10. *Suppose Assumption 4.3 holds. If  $\mu \in \mathcal{P}(E \times F)$  satisfies*

$$\mu(E \times \cdot) \ll \lambda(E \times \cdot) \quad \text{and} \quad \|\Pi_0^\mu - \Pi_0\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.},$$

*then for every  $m \in \mathbb{N}$ , we can choose a set  $C_m \in \mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0$  such that  $\mathbf{P}^\mu[\cdot \cap C_m] \ll \mathbf{P}$  on  $\mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0$  and  $\mathbf{P}^\mu(C_m^c) \rightarrow 0$  as  $m \rightarrow \infty$ .*

PROOF. By the Lebesgue decomposition theorem, we can choose for every  $m \in \mathbb{N}$  a set  $C_m \in \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0$  such that the following holds:

$$\mathbf{P}^\mu[\cdot \cap C_m] \ll \mathbf{P} \quad \text{on} \quad \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0 \quad \text{and} \quad \mathbf{P}(C_m) = 1.$$

Now note that  $\Pi_0(C_m^c) = 0$   $\mathbf{P}$ -a.s., and therefore also  $\mathbf{P}^\mu$ -a.s. as we have assumed that  $\mu(E \times \cdot) \ll \lambda(E \times \cdot)$ . Thus we obtain  $\mathbf{P}^\mu$ -a.s.

$$\Pi_0^\mu(C_m^c) = \Pi_0^\mu(C_m^c) - \Pi_0(C_m^c) \leq \|\Pi_0^\mu - \Pi_0\|_{\sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0} = \|\Pi_0^\mu - \Pi_0\|_{\mathcal{F}_{m,\infty}^0}.$$

Taking the expectation with respect to  $\mathbf{P}^\mu$  and letting  $m \rightarrow \infty$ , it follows using dominated convergence that  $\mathbf{P}^\mu(C_m^c) \rightarrow 0$  as  $m \rightarrow \infty$ .

It remains to show that  $\mathbf{P}^\mu[\cdot \cap C_m] \ll \mathbf{P}$  on the larger  $\sigma$ -field  $\mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0$ . To this end, we will establish below the following claim:  $\mathbf{P}^\mu[\cdot \cap C_m]$ -a.s.

$$\mathbf{P}^\mu[Y_{1,m-1} \in \cdot | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] \sim \mathbf{P}[Y_{1,m-1} \in \cdot | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0].$$

Let us first complete the proof assuming the claim. Let  $A \in \mathcal{F}_+^Y \vee \mathcal{F}_{m,\infty}^0$  such that  $\mathbf{P}(A) = 0$ . Then  $\mathbf{P}[A | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] = 0$   $\mathbf{P}$ -a.s., and therefore also  $\mathbf{P}^\mu[\cdot \cap C_m]$ -a.s. But then the claim implies that  $\mathbf{P}^\mu[A | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] = 0$   $\mathbf{P}^\mu[\cdot \cap C_m]$ -a.s., which yields  $\mathbf{P}^\mu(A \cap C_m) = 0$  as required.

We now proceed to prove the claim. Let  $\mathbf{R}_m^\mu$  be the probability on  $\mathcal{F}_+$  under which  $(X_k, Y_k)_{k \geq 0}$  is an inhomogeneous Markov chain with initial law  $(X_0, Y_0) \sim \mu$ , whose transition kernel is given by  $P_0 \otimes Q$  up to time  $m$  and by  $P$  after time  $m$ . Assumption 4.3 evidently implies that  $\mathbf{R}_m^\mu \sim \mathbf{P}^\mu$ , so

$$\mathbf{P}^\mu[Y_{1,m-1} \in \cdot | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] \sim \mathbf{R}_m^\mu[Y_{1,m-1} \in \cdot | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] \quad \mathbf{P}^\mu\text{-a.s.}$$

by Lemma 4.7. Now note that

$$\mathbf{R}_m^\mu[Y_{1,m-1} \in \cdot | Y_0, X_m, Y_m] = \mathbf{R}_m^\mu[Y_{1,m-1} \in \cdot | Y_0, Y_m],$$

as  $Y_{1,m}$  and  $X_m$  are conditionally independent given  $Y_0$  under  $\mathbf{R}_m^\mu$ . Therefore, we obtain using the Markov property and tower property

$$\mathbf{R}_m^\mu[Y_{1,m-1} \in \cdot | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] = \mathbf{R}_m^\mu[Y_{1,m-1} \in \cdot | Y_0, Y_m] \quad \mathbf{P}^\mu\text{-a.s.}$$

Proceeding in exactly the same manner for  $\mathbf{P}$ , we obtain

$$\begin{aligned}\mathbf{P}^\mu[Y_{1,m-1} \in \cdot | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] &\sim \mathbf{R}_m^\mu[Y_{1,m-1} \in \cdot | Y_0, Y_m] \quad \mathbf{P}^\mu[\cdot \cap C_m]\text{-a.s.}, \\ \mathbf{P}[Y_{1,m-1} \in \cdot | \sigma\{Y_0\} \vee \mathcal{F}_{m,\infty}^0] &\sim \mathbf{R}_m^\lambda[Y_{1,m-1} \in \cdot | Y_0, Y_m] \quad \mathbf{P}^\mu[\cdot \cap C_m]\text{-a.s.}\end{aligned}$$

It remains to note that  $\mathbf{R}_m^\mu[Y_{1,m-1} \in \cdot | Y_0, Y_m] = \mathbf{R}_m^\lambda[Y_{1,m-1} \in \cdot | Y_0, Y_m]$   $\mathbf{P}^\mu[\cdot \cap C_m]\text{-a.s.}$ , as  $Y_{0,m}$  is (the initial segment of) a Markov chain with transition kernel  $Q$  under both  $\mathbf{R}_m^\mu$  and  $\mathbf{R}_m^\lambda$ . Thus the proof is complete.  $\square$

The following corollary will be used a number of times.

COROLLARY 4.11. *Suppose Assumption 4.3 holds. If  $\mu$  satisfies*

$$\mu(E \times \cdot) \ll \lambda(E \times \cdot) \quad \text{and} \quad \|\Pi_0^\mu - \Pi_0\|_{\mathcal{F}_{n,\infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.},$$

*then  $\mathbf{P}^\mu|_{\mathcal{F}_+^Y} \ll \mathbf{P}|_{\mathcal{F}_+^Y}$ . If also  $\lambda(E \times \cdot) \sim \mu(E \times \cdot)$ , then  $\mathbf{P}^\mu|_{\mathcal{F}_+^Y} \sim \mathbf{P}|_{\mathcal{F}_+^Y}$ .*

PROOF. Define the sets  $C_m$  as in Lemma 4.10, and choose any  $A \in \mathcal{F}_+^Y$  with  $\mathbf{P}(A) = 0$ . Then  $\mathbf{P}^\mu(A) = \mathbf{P}^\mu(A \cap C_m^c) \leq \mathbf{P}^\mu(C_m^c) \rightarrow 0$  as  $m \rightarrow \infty$ . This yields the first claim. On the other hand, note that the proof of Lemma 4.10 does not use the invariance of  $\lambda$ . Thus we may exchange the roles of  $\mu$  and  $\lambda$  in Lemma 4.10 to obtain also  $\mathbf{P}|_{\mathcal{F}_+^Y} \ll \mathbf{P}^\mu|_{\mathcal{F}_+^Y}$  when  $\lambda(E \times \cdot) \sim \mu(E \times \cdot)$ .  $\square$

We can now complete the proof of Theorem 4.4.

PROOF OF THEOREM 4.4. Fix  $\mu$  as in the statement of the Theorem, and define the corresponding sets  $C_m$  as in Lemma 4.10. Let  $\mathbf{P}_m^\mu = \mathbf{P}^\mu[\cdot | C_m]$  and  $\mathbf{P}_m^{\mu^\perp} = \mathbf{P}^\mu[\cdot | C_m^c]$ . By the Bayes formula, we can write for any  $A \in \mathcal{F}^+$

$$\mathbf{P}^\mu[A | \mathcal{F}_{0,n}^Y] = \mathbf{P}_m^\mu[A | \mathcal{F}_{0,n}^Y] \mathbf{P}^\mu[C_m | \mathcal{F}_{0,n}^Y] + \mathbf{P}_m^{\mu^\perp}[A | \mathcal{F}_{0,n}^Y] \mathbf{P}^\mu[C_m^c | \mathcal{F}_{0,n}^Y] \quad \mathbf{P}^\mu\text{-a.s.}$$

In particular, if we define  $\Sigma_n^m = \mathbf{P}_m^\mu[\cdot | \mathcal{F}_{0,n}^Y]$ , we can write

$$|\Pi_n^\mu(A) - \Pi_n(A)| \leq |\Sigma_n^m(A) - \Pi_n(A)| \Pi_n^\mu(C_m) + \Pi_n^\mu(C_m^c) \quad \mathbf{P}^\mu\text{-a.s.}$$

As  $\mathcal{F}_+$  is countably generated, the  $\mathbf{P}^\mu$ -exceptional set can be chosen independent of  $A$  using a monotone class argument. We therefore obtain

$$\mathbf{E}^\mu \left[ \limsup_{n \rightarrow \infty} \|\Pi_n^\mu - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \right] \leq \mathbf{E}_m^\mu \left[ \limsup_{n \rightarrow \infty} \|\Sigma_n^m - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \right] + \mathbf{P}^\mu(C_m^c).$$

But the first term on the right vanishes by Lemma 4.9. Therefore, using that  $\mathbf{P}^\mu(C_m^c) \rightarrow 0$  as  $m \rightarrow \infty$ , we find that  $\|\Pi_n^\mu - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^0} \rightarrow 0$   $\mathbf{P}^\mu\text{-a.s.}$

This completes the proof when  $\lambda(E \times \cdot) \ll \mu(E \times \cdot)$ . To conclude, note that  $\mathbf{P}$ -a.s. convergence follows by Corollary 4.11 when  $\lambda(E \times \cdot) \sim \mu(E \times \cdot)$ .  $\square$

4.3. *Filter stability and ergodicity.* Using the local stability Theorem 4.4 we can now proceed to obtain filter stability results that are applicable to infinite-dimensional or weak-\* ergodic models, in analogy with the ergodic results obtained in section 2. While many variations on these results are possible, we give two representative results that suffice in all the examples that will be given in section 5 below. Beside stability, we will also consider following Kunita [21] the ergodic properties of the filtering process  $(\pi_n)_{n \geq 0}$  when it is considered as a measure-valued Markov process.

4.3.1. *Filter stability and local mixing.* In this short subsection, we assume that the state space  $E$  of the unobserved process is contained in a countable product  $E \subseteq \prod_{i \in I} E^i$ , where each  $E^i$  is Polish. We are therefore in the local mixing setting of section 2.2. In the present section, we define

$$\mathcal{F}_{m,n}^J = \sigma\{X_{m,n}^J, Y_{m,n}\} \quad \text{for } J \subseteq I, \quad m \leq n,$$

that is, we include the observations in the local filtration. We also denote by  $\mathcal{E}^J \subseteq \mathcal{B}(E)$  the cylinder  $\sigma$ -field generated by the coordinates in  $J \subseteq I$ .

The bivariate Markov chain  $(X_n, Y_n)_{n \geq 0}$  is said to be *locally mixing* if

$$\|\mathbf{P}^{x,y} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^J} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } (x,y) \in E \times F \text{ and } J \subseteq I, \quad |J| < \infty.$$

It is easily seen that this coincides with the notion introduced in section 2.2. The following filter stability result follows trivially from Corollary 4.6.

COROLLARY 4.12. *Suppose that the Markov chain  $(X_n, Y_n)_{n \geq 0}$  is locally mixing and that Assumption 4.3 holds. Then for any  $\mu \in \mathcal{P}(E \times F)$  such that  $\mu(E \times \cdot) \ll \lambda(E \times \cdot)$ , and for any  $r \in \mathbb{N}$ , we have*

$$\|\Pi_n^\mu - \Pi_n\|_{\mathcal{F}_{n-r,\infty}^J} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.} \quad \text{for all } J \subseteq I, \quad |J| < \infty.$$

In particular, the filter is stable in the sense

$$\|\pi_n^\mu - \pi_n\|_{\mathcal{E}^J} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s.} \quad \text{for all } J \subseteq I, \quad |J| < \infty.$$

If  $\mu(E \times \cdot) \sim \lambda(E \times \cdot)$ , the convergence holds also  $\mathbf{P}$ -a.s.

4.3.2. *Filter stability and asymptotic coupling.* The goal of this section is to develop a filter stability counterpart of the weak-\* ergodic theorem in section 2.4. For sake of transparency, we will restrict attention to a special class of bivariate Markov chains, known as hidden Markov models, that arise in many settings (cf. section 5). While our method is certainly also applicable

in more general situations, the hidden Markov assumption will allow us to state concrete and easily verifiable conditions for weak-\* filter stability.

A *hidden Markov model* is a bivariate Markov chain  $(X_k, Y_k)_{k \geq 0}$  (in the Polish state space  $E \times F$ ) whose transition kernel  $P$  factorizes as

$$P(x, y, dx', dy') = P_0(x, dx') \Phi(x', dy')$$

for transition kernels  $P_0 : E \times \mathcal{B}(E) \rightarrow [0, 1]$  and  $\Phi : E \times \mathcal{B}(F) \rightarrow [0, 1]$ . The special feature of such models is that the unobserved process  $(X_k)_{k \geq 0}$  is a Markov chain in its own right, and the observations  $(Y_k)_{k \geq 0}$  are conditionally independent given  $(X_k)_{k \geq 0}$ . This is a common scenario when  $(Y_k)_{k \geq 0}$  represent noisy observations of an underlying Markov chain  $(X_k)_{k \geq 0}$ . In this setting, it is natural to consider just initial conditions for  $X_0$ , rather than for the pair  $(X_0, Y_0)$ . We therefore define  $\mathbf{P}^x = \mathbf{P}^{\delta_x \otimes \Phi(x, \cdot)}$  for  $x \in E$  and  $\mathbf{P}^\mu = \int \mathbf{P}^x \mu(dx)$  for  $\mu \in \mathcal{P}(E)$ , as well as the corresponding filters  $\pi_n^\mu, \Pi_n^\mu$ . We will assume that  $P_0$  admits an invariant probability  $\hat{\lambda} \in \mathcal{P}(E)$ , so that  $\lambda = \hat{\lambda} \otimes \Phi$  is invariant for  $(X_n, Y_n)_{n \geq 0}$  (this entails no loss of generality if we assume, as we do, that  $(X_n, Y_n)_{n \geq 0}$  admits an invariant probability).

A hidden Markov model is called *nondegenerate* if the observation kernel  $\Phi$  admits a positive density with respect to some reference measure  $\varphi$  on  $F$ :

$$\Phi(x, dy) = g(x, y) \varphi(dy), \quad g(x, y) > 0 \text{ for all } (x, y) \in E \times F.$$

Evidently, nondegeneracy of the hidden Markov model corresponds to the validity of Assumption 4.3 for the bivariate Markov chain  $(X_k, Y_k)_{k \geq 0}$ .

We can now state our weak-\* stability result for the filter in the hidden Markov model setting; compare with the weak-\* ergodic Theorem 2.10. In the following, we fix a complete metric  $d$  for the Polish space  $E$ . To allow for the case that the observation law is discontinuous with respect to  $d$  (see section 5 for examples), we introduce an auxiliary quantity  $\tilde{d}$  that dominates the metric  $d$ . Let us note that it is not necessary for  $\tilde{d}$  to be a metric.

**THEOREM 4.13** (Weak-\* filter stability). *Let  $(X_k, Y_k)_{k \geq 0}$  be a nondegenerate hidden Markov model that admits an invariant probability  $\hat{\lambda}$ , and let  $\tilde{d}(x, y) \geq d(x, y)$  for all  $x, y \in E$ . Suppose that the following hold:*

a. (Asymptotic coupling) *There exists  $\alpha > 0$  such that*

$$\forall x, x' \in E \quad \exists \mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'}) \quad \text{s.t.} \quad \mathbf{Q} \left[ \sum_{n=1}^{\infty} \tilde{d}(X_n, X'_n)^2 < \infty \right] \geq \alpha.$$

b. (*Hellinger-Lipschitz observations*) There exists  $C < \infty$  such that

$$\int \{\sqrt{g(x, y)} - \sqrt{g(x', y)}\}^2 \varphi(dy) \leq C \tilde{d}(x, x')^2 \quad \text{for all } x, x' \in E.$$

Then the filter is stable in the sense that

$$|\pi_n^\mu(f) - \pi_n^\nu(f)| \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\gamma\text{-a.s.} \quad \text{for all } f \in \text{Lip}(E), \quad \mu, \nu, \gamma \in \mathcal{P}(E).$$

In particular, we obtain

$$\|\pi_n^\mu - \pi_n^\nu\|_{\text{BL}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^\gamma\text{-probability} \quad \text{for all } \mu, \nu, \gamma \in \mathcal{P}(E).$$

PROOF. The proof is similar to that of Theorem 2.10. Let  $(\xi_n)_{n \geq 0}$  be an i.i.d. sequence of standard Gaussian random variables independent of  $(X_n, Y_n)_{n \geq 0}$ , so that we may consider in the following the extended Markov chain  $(X_n, \xi_n, Y_n)_{n \geq 0}$ . Fix  $f \in \text{Lip}(E)$ , and define  $F_n = f(X_n) + \xi_n$ . Conditionally on  $\mathcal{F}_+^X$ , the process  $(F_n, Y_n)_{n \geq 0}$  is an independent sequence with

$$\begin{aligned} \mathbf{P}^\mu[(F_n, Y_n)_{n \geq k} \in A | \mathcal{F}_+^X] &= \mathbf{R}^{X_k, \infty}(A) := \\ &\int \mathbf{1}_A(r, y) \prod_{n=k}^{\infty} \frac{e^{-(r_n - f(X_n))^2/2}}{\sqrt{2\pi}} dr_n g(X_n, y_n) \varphi(dy_n), \quad A \in \mathcal{B}(\mathbb{R} \times F)^{\otimes \mathbb{N}} \end{aligned}$$

for all  $\mu \in \mathcal{P}(E)$ . We can now estimate as in the proof of Lemma 2.11

$$\begin{aligned} \|\mathbf{R}^{x_0, \infty} - \mathbf{R}^{x'_0, \infty}\|^2 &\leq \sum_{n=0}^{\infty} \left[ 2\{f(x_n) - f(x'_n)\}^2 + 8C \tilde{d}(x_n, x'_n)^2 \right] \\ &\leq (8C + 2) \sum_{n=0}^{\infty} \tilde{d}(x_n, x'_n)^2, \end{aligned}$$

where we have used that  $1 - \prod_n (1 - p_n) \leq \sum_n p_n$  when  $0 \leq p_n \leq 1$  for all  $n$ . Therefore, proceeding exactly as in the proof of Theorem 2.10, we find that for every  $x, x' \in E$ , there exists  $n \geq 1$  such that we have

$$\|\mathbf{P}^x[F_{n, \infty}, Y_{n, \infty} \in \cdot] - \mathbf{P}^{x'}[F_{n, \infty}, Y_{n, \infty} \in \cdot]\| \leq 2 - \alpha.$$

Now note that the law  $\mathbf{P}^x[F_{n, \infty}, Y_{n, \infty} \in \cdot]$  does not change if we condition additionally on  $\xi_0, Y_0$  (as  $\xi_0, Y_0$  are independent of  $X_{1, \infty}, \xi_{1, \infty}, Y_{1, \infty}$  under  $\mathbf{P}^x$ ). We can therefore apply Theorem 2.1 to conclude that

$$\|\mathbf{P}^\mu[F_{n, \infty}, Y_{n, \infty} \in \cdot] - \mathbf{P}^\nu[F_{n, \infty}, Y_{n, \infty} \in \cdot]\| \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } \mu, \nu \in \mathcal{P}(E).$$



Moreover, note that  $\mathbf{P}^\mu[Y_0 \in \cdot] \sim \varphi$  for all  $\mu \in \mathcal{P}(E)$ . Thus we can apply Corollary 4.6 to conclude (here  $\xi \in \mathcal{P}(\mathbb{R})$  is the standard Gaussian measure)

$$\|\pi_n^\mu f^{-1} * \xi - \pi_n f^{-1} * \xi\| \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s. and } \mathbf{P}\text{-a.s.}$$

Applying the argument in the proof of Theorem 2.10 pathwise, we obtain

$$|\pi_n^\mu(f) - \pi_n(f)| \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^\mu\text{-a.s. and } \mathbf{P}\text{-a.s.}$$

Thus by the triangle inequality, we have

$$|\pi_n^\mu(f) - \pi_n^\nu(f)| \leq |\pi_n^\mu(f) - \pi_n(f)| + |\pi_n^\nu(f) - \pi_n(f)| \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}\text{-a.s.}$$

Finally, note that the assumptions of Corollary 4.11 are satisfied for any initial measure, so that  $\mathbf{P}^\gamma|_{\mathcal{G}_+^Y} \ll \mathbf{P}|_{\mathcal{G}_+^Y}$  for any  $\gamma \in \mathcal{P}(E)$ . Thus the above  $\mathbf{P}$ -a.s. convergence also holds  $\mathbf{P}^\gamma$ -a.s., which yields the first conclusion.

To obtain the second conclusion, we argue as in the proof of Theorem 2.12. Fix  $\varepsilon > 0$ . Let  $K \subseteq E$  be a compact set such that  $\lambda(K) \geq 1 - \varepsilon$ , and define  $\chi(x) = (1 - \varepsilon^{-1}d(x, K))_+$ . Following the argument used in the proof of Theorem 2.12, we can find functions  $f_1, \dots, f_k \in \text{Lip}(E)$  such that

$$\|\pi_n^\mu - \pi_n\|_{\text{BL}} \leq \max_{i=1, \dots, k} |\pi_n^\mu(f_i \chi) - \pi_n(f_i \chi)| + |\pi_n^\mu(\chi) - \pi_n(\chi)| + 2\pi_n(K^c) + 6\varepsilon$$

for all  $n \geq 0$ . Taking the expectation and letting  $n \rightarrow \infty$ , we obtain

$$\limsup_{n \rightarrow \infty} \mathbf{E}[\|\pi_n^\mu - \pi_n\|_{\text{BL}}] \leq 8\varepsilon.$$

As  $\varepsilon > 0$  is arbitrary, this implies that

$$\|\pi_n^\mu - \pi_n\|_{\text{BL}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } \mathbf{P}\text{-probability.}$$

But  $\mathbf{P}^\gamma|_{\mathcal{G}_+^Y} \ll \mathbf{P}|_{\mathcal{G}_+^Y}$ , so the convergence is also in  $\mathbf{P}^\gamma$ -probability. Applying the triangle inequality and dominated convergence completes the proof.  $\square$

**REMARK 4.14.** In Theorem 4.13, we obtain a.s. stability of the filter for individual Lipschitz functions, but only stability in probability for the  $\|\cdot\|_{\text{BL}}$  norm. It is not clear whether the latter could be improved to a.s. convergence (except when  $E$  is compact, in which case the compactness argument used in the proof above directly yields a.s. convergence). The problem is that we do not know whether the null set in the a.s. stability result can be made independent of the choice of Lipschitz function; if this were the case, the method used in Theorem 2.10 could be used to obtain a.s. convergence.

**4.3.3. Ergodicity of the filter.** We developed above a number of filter stability results that ensure convergence of conditional expectations of the form  $|\mathbf{E}^\mu[f(X_n)|\mathcal{F}_{0,n}^Y] - \mathbf{E}[f(X_n)|\mathcal{F}_{0,n}^Y]| \rightarrow 0$ . This can evidently be viewed as a natural conditional counterpart to the classical ergodic theory of Markov chains, which ensures that  $|\mathbf{E}^\mu[f(X_n)] - \mathbf{E}[f(X_n)]| \rightarrow 0$ . In this section, following Kunita [21], we develop a different ergodic property of the filter.

It is well known—and a simple exercise using the Bayes formula—that the filter  $\pi_n^\mu$  can be computed in a recursive fashion. In particular, under the nondegeneracy Assumption 4.3, we have  $\pi_{n+1}^\mu = U(\pi_n^\mu, Y_n, Y_{n+1})$  with

$$U(\nu, y, y')(A) = \frac{\int \mathbf{1}_A(x') g(x, y, x', y') P_0(x, dx') \nu(dx)}{\int g(x, y, x', y') P_0(x, dx') \nu(dx)}.$$

Let  $H : \mathcal{P}(E) \times F \rightarrow \mathbb{R}$  be a bounded measurable function. Then

$$\mathbf{E}^\mu[H(\pi_{n+1}^\mu, Y_{n+1})|\mathcal{F}_{0,n}^Y] = \mathbf{E}^\mu[H(U(\pi_n^\mu, Y_n, Y_{n+1}), Y_{n+1})|\mathcal{F}_{0,n}^Y] = \Gamma H(\pi_n^\mu, Y_n),$$

where the kernel  $\Gamma : \mathcal{P}(E) \times F \times \mathcal{B}(\mathcal{P}(E) \times F) \rightarrow [0, 1]$  is given by

$$\Gamma(\nu, y, A) = \int \mathbf{1}_A(U(\nu, y, y'), y') P(x, y, dx', dy') \nu(dx).$$

Thus we see that the process  $(\pi_n^\mu, Y_n)_{n \geq 0}$  is itself a  $(\mathcal{P}(E) \times F)$ -valued Markov chain under  $\mathbf{P}^\mu$  with transition kernel  $\Gamma$ . In the hidden Markov model setting of section 4.3.2,  $\Gamma(\nu, y, A)$  does not depend on  $y$ , so that in this special case even the filter  $(\pi_n^\mu)_{n \geq 0}$  itself is a  $\mathcal{P}(E)$ -valued Markov chain.

In view of this Markov property of the filter, it is now natural to ask about the ergodic properties of the filtering process itself. Generally speaking, we would like to know whether the ergodic properties of the underlying Markov chain  $(X_n, Y_n)_{n \geq 0}$  with transition kernel  $P$  are “lifted” to the measure-valued Markov chain  $(\pi_n^\mu, Y_n)_{n \geq 0}$  with transition kernel  $\Gamma$ . Following the seminal work of Kunita [21], such questions have been considered by a number of authors [37, 3, 5, 43, 39]. We will focus here on the question of unique ergodicity, where the following result (essentially due to Kunita) is known.

**THEOREM 4.15.** *Suppose that Assumption 4.3 holds, and that the transition kernel  $P$  admits a unique invariant measure  $\lambda \in \mathcal{P}(E \times F)$ . Then the transition kernel  $\Gamma$  admits a unique invariant measure  $\Lambda \in \mathcal{P}(\mathcal{P}(E) \times F)$  iff*

$$\bigcap_{n \geq 0} \mathcal{F}_-^Y \vee \mathcal{F}_{-\infty, -n}^X = \mathcal{F}_-^Y \quad \text{mod } \mathbf{P}.$$

We refer to [43] for a full proof in the hidden Markov model setting, which is easily adapted to the more general setting considered here (sufficiency is also shown in our setting in the proof of [39, Theorem 2.12]).

To prove unique ergodicity of the filter, we must therefore establish the measure-theoretic identity in Theorem 4.15. The goal of this section is to accomplish this task under the same assumptions we have used for filter stability: local mixing or asymptotic couplings. In fact, slightly weaker forms of the assumptions of Corollary 4.12 or Theorem 4.13 will suffice. We refer to sections 4.3.1 and 4.3.2 for the notation used in the following results.

**THEOREM 4.16.** *Suppose that Assumption 4.3 holds. Let  $(X_n, Y_n)_{n \geq 0}$  with  $E \subseteq \prod_{i \in I} E^i$  be uniquely ergodic and a.e. locally mixing in the sense*

$$\|\mathbf{P}^{x,y} - \mathbf{P}\|_{\mathcal{F}_{n,\infty}^J} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \lambda\text{-a.e. } (x,y) \text{ and all } J \subseteq I, |J| < \infty.$$

*Then the filter transition kernel  $\Gamma$  admits a unique invariant measure.*

**THEOREM 4.17.** *Let  $(X_k, Y_k)_{k \geq 0}$  be a hidden Markov model that is non-degenerate and that admits a unique invariant probability  $\hat{\lambda}$ . Moreover, let  $\tilde{d}(x, y) \geq d(x, y)$  for all  $x, y \in E$ , and suppose that the following hold:*

*a. (Asymptotic coupling) For  $\hat{\lambda} \otimes \hat{\lambda}$ -a.e.  $(x, x') \in E \times E$ ,*

$$\exists \mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'}) \quad \text{such that} \quad \mathbf{Q} \left[ \sum_{n=1}^{\infty} \tilde{d}(X_n, X'_n)^2 < \infty \right] > 0.$$

*b. (Hellinger-Lipschitz observations) There exists  $C < \infty$  such that*

$$\int \{ \sqrt{g(x, y)} - \sqrt{g(x', y)} \}^2 \varphi(dy) \leq C \tilde{d}(x, x')^2 \quad \text{for all } x, x' \in E.$$

*Then the filter transition kernel  $\Gamma$  admits a unique invariant measure.*

The remainder of this section is devoted to the proof of these results. We must begin by obtaining a local result in the setting of section 4.2. To this end, we will need the following structure result for the invariant measure  $\lambda$ .

**LEMMA 4.18.** *Suppose that Assumptions 4.2 and 4.3 hold. Then the invariant measure  $\lambda$  satisfies  $\lambda(dx, dy) \sim \lambda(dx \times F) \otimes \lambda(E \times dy)$ .*

**PROOF.** Assumption 4.2 and Jensen's inequality yield  $\mathbf{P}$ -a.s.

$$\mathbf{E}[|\mathbf{P}[Y_n \in \cdot | Y_0] - \lambda(E \times \cdot)|] \leq \mathbf{E}[|\mathbf{P}^{X_0, Y_0}[Y_n \in \cdot] - \lambda(E \times \cdot)|] \xrightarrow{n \rightarrow \infty} 0.$$

The result follows by applying [39, Prop. 3.3] to the process  $(Y_n, X_n)_{n \geq 0}$ .  $\square$

We can now prove the following local result.

COROLLARY 4.19. *Suppose that Assumptions 4.2 and 4.3 hold. Then*

$$\mathbf{P} \left[ A \middle| \bigcap_{n \geq 0} \mathcal{F}_-^Y \vee \mathcal{F}_{-\infty, -n}^X \right] = \mathbf{P}[A | \mathcal{F}_-^Y] \quad \mathbf{P}\text{-a.s.} \quad \text{for every } A \in \mathcal{F}_{-\infty, \infty}^0.$$

PROOF. By Lemma 4.18, we have  $\lambda(dx, dy) = h(x, y) \lambda(dx \times F) \lambda(E \times dy)$  for a strictly positive measurable function  $h$ . Define the kernel

$$\lambda_x(A) = \frac{\int \mathbf{1}_A(y) h(x, y) \lambda(E \times dy)}{\int h(x, y) \lambda(E \times dy)}.$$

By the Bayes formula  $\lambda_{X_0} = \mathbf{P}[Y_0 \in \cdot | X_0]$ . Assumption 4.2, disintegration, and Jensen's inequality yield a set  $H \subseteq E$  with  $\lambda(H \times F) = 1$  such that

$$\mathbf{E}^{\delta_x \otimes \lambda_x} [\|\mathbf{P}^{x, Y_0} - \mathbf{P}\|_{\mathcal{F}_{n, \infty}^0}] \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \mathbf{E}^{\delta_x \otimes \lambda_x} [\|\Pi_0 - \mathbf{P}\|_{\mathcal{F}_{n, \infty}^0}] \xrightarrow{n \rightarrow \infty} 0$$

for all  $x \in H$ . But note that  $\Pi_0^{\delta_x \otimes \lambda_x} = \mathbf{P}^{x, Y_0}$  holds  $\mathbf{P}^{\delta_x \otimes \lambda_x}$ -a.s. Thus

$$\lambda_x \sim \lambda(E \times \cdot) \quad \text{and} \quad \|\Pi_0^{\delta_x \otimes \lambda_x} - \Pi_0\|_{\mathcal{F}_{n, \infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^{\delta_x \otimes \lambda_x}\text{-a.s.}$$

for every  $x \in H$  by the definition of  $\lambda_x$  and the triangle inequality (we have used that  $\|\Pi_0^{\delta_x \otimes \lambda_x} - \Pi_0\|_{\mathcal{F}_{n, \infty}^0}$  is pointwise decreasing to establish a.s. convergence). Therefore, by Theorem 4.4, we obtain for every  $x \in H$

$$\|\Pi_n^{\delta_x \otimes \lambda_x} - \Pi_n\|_{\mathcal{F}_{n-r, \infty}^0} \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}^{\delta_x \otimes \lambda_x}\text{-a.s.} \quad \text{for any } r \in \mathbb{N}.$$

Now note that  $\Pi_n^{\delta_{X_0} \otimes \lambda_{X_0}}(A) = \mathbf{P}[A | \sigma X_0 \vee \mathcal{F}_{0, n}^Y]$  for all  $A \in \mathcal{F}_+$ , for example, by Lemma 3.11. It follows that for any  $A \in \mathcal{F}_{0, \infty}^0$ , we have the convergence

$$\mathbf{E}[\|\mathbf{E}[\mathbf{1}_A \circ \Theta^{n-r} | \sigma X_0 \vee \mathcal{F}_{0, n}^Y] - \mathbf{E}[\mathbf{1}_A \circ \Theta^{n-r} | \mathcal{F}_{0, n}^Y]\|] \xrightarrow{n \rightarrow \infty} 0.$$

But the Markov property of  $(X_n, Y_n)_{n \in \mathbb{Z}}$  yields  $\mathbf{E}[\mathbf{1}_A \circ \Theta^{n-r} | \sigma X_0 \vee \mathcal{F}_{0, n}^Y] = \mathbf{E}[\mathbf{1}_A \circ \Theta^{n-r} | \mathcal{F}_{-\infty, 0}^X \vee \mathcal{F}_{-\infty, n}^Y]$ . Therefore, using stationarity, we obtain

$$\mathbf{E}[\|\mathbf{E}[\mathbf{1}_A \circ \Theta^{-r} | \mathcal{F}_-^Y \vee \mathcal{F}_{-\infty, -n}^X] - \mathbf{E}[\mathbf{1}_A \circ \Theta^{-r} | \mathcal{F}_{-n, 0}^Y]\|] \xrightarrow{n \rightarrow \infty} 0.$$

The lemma now follows by the martingale convergence theorem for  $A \in \mathcal{F}_{-r, \infty}^0$ . As  $r$  is arbitrary, a monotone class argument concludes the proof.  $\square$

The proof of Theorem 4.16 is now essentially trivial.

PROOF OF THEOREM 4.16. By Corollary 4.19, we have

$$\mathbf{P} \left[ A \middle| \bigcap_{n \geq 0} \mathcal{F}_-^Y \vee \mathcal{F}_{-\infty, -n}^X \right] = \mathbf{P}[A | \mathcal{F}_-^Y] \quad \mathbf{P}\text{-a.s.} \quad \text{for every } A \in \mathcal{F}_{-\infty, \infty}^J$$

whenever  $J \subseteq I$ ,  $|J| < \infty$ . A monotone class argument yields the conclusion for all  $A \in \mathcal{F}$ . Thus the  $\sigma$ -field identity of Theorem 4.15 holds.  $\square$

We now turn to the proof of Theorem 4.17.

PROOF OF THEOREM 4.17. Proceeding precisely as in the proof of Theorem 4.13 (and adopting the same notation as is used there), we find that for  $\hat{\lambda} \otimes \hat{\lambda}$ -a.e.  $(x, x') \in E \times E$ , there exists  $n \geq 1$  such that

$$\|\mathbf{P}^x[F_{n, \infty}, Y_{n, \infty} \in \cdot] - \mathbf{P}^{x'}[F_{n, \infty}, Y_{n, \infty} \in \cdot]\| < 2.$$

From Theorem 2.2, it follows that

$$\|\mathbf{P}^{x, y}[F_{n, \infty}, Y_{n, \infty} \in \cdot] - \mathbf{P}[F_{n, \infty}, Y_{n, \infty} \in \cdot]\| \xrightarrow{n \rightarrow \infty} 0 \quad \lambda\text{-a.e. } (x, y) \in E \times F.$$

Applying Corollary 4.19, we find that

$$\mathbf{P} \left[ A \middle| \bigcap_{n \geq 0} \mathcal{F}_-^Y \vee \mathcal{F}_{-\infty, -n}^{X, \xi} \right] = \mathbf{P}[A | \mathcal{F}_-^Y] \quad \mathbf{P}\text{-a.s.} \quad \text{for every } A \in \mathcal{F}_{-\infty, \infty}^{F, Y},$$

where  $\mathcal{F}_{m, n}^{X, \xi} = \sigma\{X_{m, n}, \xi_{m, n}\}$  and  $\mathcal{F}_{m, n}^{F, Y} = \sigma\{F_{m, n}, Y_{m, n}\}$ . In particular, if

$$G = g(f(X_{-m}) + \xi_{-m}, \dots, f(X_m) + \xi_m, Y_{-m}, \dots, Y_m)$$

for some bounded continuous function  $g : \mathbb{R}^{2m+1} \times F^{2m+1} \rightarrow \mathbb{R}$ , then

$$\mathbf{E} \left[ G \middle| \bigcap_{n \geq 0} \mathcal{F}_-^Y \vee \mathcal{F}_{-\infty, -n}^X \right] = \mathbf{E}[G | \mathcal{F}_-^Y] \quad \mathbf{P}\text{-a.s.}$$

as  $\sigma\{\xi_{-\infty, -n}\}$  is independent of  $\mathcal{F}^X \vee \mathcal{F}^Y \vee \sigma\{\xi_{-m, m}\}$  for  $n \geq m$ . Now note that nothing in the proof relied on the fact that  $\xi_k$  are Gaussian with unit variance; we can replace  $\xi_k$  by  $\varepsilon \xi_k$  for any  $\varepsilon > 0$  and attain the same conclusion. Letting  $\varepsilon \rightarrow 0$ , we find that for any  $f \in \text{Lip}(E)$ ,  $m \geq 0$ , and bounded continuous  $g : \mathbb{R}^{2m+1} \times F^{2m+1} \rightarrow \mathbb{R}$ , the above identity holds for

$$G = g(f(X_{-m}), \dots, f(X_m), Y_{-m}, \dots, Y_m).$$

The remainder of the proof is a routine approximation argument. As  $E$  is Polish, we can choose a countable dense subset  $E' \subset E$ . Then the countable family of open balls  $\{B(x, \delta) : x \in E', \delta \in \mathbb{Q}_+\}$  (where  $B(x, \delta)$  is the open ball with center  $x$  and radius  $\delta$ ) generate the Borel  $\sigma$ -field  $\mathcal{B}(E)$ . Arrange these open balls arbitrarily as a sequence  $(B_k)_{k \geq 1}$ , and define the functions

$$\iota(x) = \sum_{k=1}^{\infty} \frac{\mathbf{1}_{B_k}(x)}{3^k}, \quad \iota_r^\delta(x) = \sum_{k=1}^r \frac{\delta^{-1} d(x, B_k^c) \wedge 1}{3^k} \quad (x \in E).$$

Then  $\iota_r^\delta$  is bounded and Lipschitz for every  $r, \delta$ , and  $\iota_r^\delta \rightarrow \iota$  as  $\delta \downarrow 0, r \uparrow \infty$ . Choosing  $f = \iota_r^\delta$  and taking limits, we obtain the above identity for

$$G = g(\iota(X_{-m}), \dots, \iota(X_m), Y_{-m}, \dots, Y_m)$$

for any  $m \geq 0$  and bounded continuous  $g : \mathbb{R}^{2m+1} \times F^{2m+1} \rightarrow \mathbb{R}$ . A monotone class argument shows that we may choose  $G$  to be any bounded  $\sigma\{\iota(X_k), Y_k : k \in \mathbb{Z}\}$ -measurable function. But  $\mathcal{B}(E) = \sigma\{\iota\}$ , so the proof is complete.  $\square$

**4.4. Continuous time.** Up to this point we have considered only discrete-time processes and Markov chains. However, continuous time processes are of equal interest in many applications: indeed, most of the examples that we will consider in section 5 will be in continuous time. The goal of this section is to extend our main filter stability results to the continuous time setting.

In principle, we can view continuous time processes as a special case of the discrete time setting. If  $(x_t, y_t)_{t \geq 0}$  is a continuous time Markov process with càdlàg paths, then we can define the associated discrete-time Markov chain  $(X_n, Y_n)_{n \geq 0}$  with values in the Skorokhod space  $D([0, 1]; E \times F)$  by setting  $X_n = (x_t)_{t \in [n, n+1]}$  and  $Y_n = (y_t)_{t \in [n, n+1]}$ . When we are dealing with the (unconditional) ergodic theory of  $(x_t, y_t)_{t \geq 0}$ , we can obtain continuous-time ergodic results directly from the corresponding results for the discrete-time chain  $(X_n, Y_n)_{n \geq 0}$ . However, in the conditional setting, two issues arise.

First, note that in the unconditional setting, the marginal law  $\mathbf{P}[x_t \in \cdot]$  for  $t \in [n, n+1]$  is a coordinate projection of  $\mathbf{P}[X_n \in \cdot]$ . However, this is not true for the filter: the projection of  $\mathbf{P}[X_n \in \cdot | \mathcal{F}_{0,n}^Y]$  gives  $\mathbf{P}[x_t \in \cdot | (y_s)_{s \in [0, n+1]}]$ , not the continuous time filter  $\pi_t = \mathbf{P}[x_t \in \cdot | (y_s)_{s \in [0, t]}]$ . We must therefore get rid of the additional observation segment  $(y_s)_{s \in [t, n+1]}$  that appears in the projection. This is precisely what will be done in this section.

Second, in continuous time, numerous subtleties arise in defining the filtering process  $(\pi_t)_{t \geq 0}$  as a stochastic process with sufficiently regular sample paths. Such issues would have to be dealt with carefully if we wanted to obtain, for example, almost sure filter stability results in the continuous time

setting. The structure of nonlinear filters in continuous time is a classical topic in stochastic analysis (see, for example, [25, 47, 22]) that provides the necessary tools to address such problems. However, in the present setting, such regularity issues are purely technical in nature and do not introduce any new ideas in the ergodic theory of nonlinear filters. We therefore choose to circumvent these issues by considering only stability in probability in the continuous time setting, in which case regularity issues can be avoided.

A final issue that arises in the continuous time setting is that, unlike in discrete time, one must make a distinction between general bivariate Markov processes and hidden Markov processes, as we will presently explain.

Recall that a discrete-time hidden Markov model is defined by the fact that  $(X_n)_{n \geq 0}$  is itself Markov and  $(Y_n)_{n \geq 0}$  are conditionally independent given  $(X_n)_{n \geq 0}$ . In continuous time, we cannot assign a (conditionally) independent random variable to every time  $t \in \mathbb{R}_+$ . Instead, we consider an integrated form of the observations where  $(x_t)_{t \geq 0}$  is a Markov process and  $(y_t)_{t \geq 0}$  has conditionally independent increments given  $(x_t)_{t \geq 0}$ . This is known as a Markov additive process [6], and constitutes the natural continuous-time counterpart to a hidden Markov model [47]. For example, the most common observation model in continuous time is the “white noise” model [25]

$$y_t = \int_0^t h(x_s) ds + W_t,$$

where  $(W_t)_{t \geq 0}$  is a Brownian motion independent of  $(x_t)_{t \geq 0}$ . Formally,  $dy_t/dt$  represents the observation of  $h(x_t)$  corrupted by white noise, but the integrated form is used to define a mathematically sensible model. In this example, the pair  $(x_t, y_t)_{t \geq 0}$  is evidently a Markov additive process.

In principle, a continuous-time hidden Markov process is a special case of a bivariate Markov process as in the discrete time setting. Unfortunately, as  $y_t$  is an additive process, it cannot be positive recurrent except in trivial cases, so the pair  $(x_t, y_t)_{t \geq 0}$  does not admit an invariant probability. We must therefore take care to utilize explicitly the fact that it is the increments of  $y_t$ , and not  $y_t$  itself, that will be stationary under the invariant distribution. This does not introduce any complications into our theory: both the bivariate Markov setting and the Markov additive setting can be treated in exactly the same manner. However, two distinct sets of notation are required for these two settings. In order to avoid notational confusion, we will develop our continuous time results below in the hidden Markov process setting only (all our examples in section 5 will be of this form). The same approach can however be adapted to the bivariate Markov setting with minimal effort.

4.4.1. *The continuous time setting.* In the remainder of this section, we consider a continuous-time process  $(x_t, y_t)_{t \geq 0}$  with càdlàg paths, where  $x_t$  takes values in a Polish space  $E$  and  $y_t$  takes values in a Polish topological vector space  $F$ . We realize this process on the canonical path space  $\Omega = D(\mathbb{R}_+; E \times F)$  endowed with its Borel  $\sigma$ -field  $\mathcal{F}$ , such that  $x_t(\xi, \eta) = \xi(t)$  and  $y_t(\xi, \eta) = \eta(t)$ . We define for  $s \leq t$  the  $D([0, t-s]; E)$ -valued random variable  $x_{s,t} = (x_r)_{r \in [s,t]}$  and the  $\sigma$ -field  $\mathcal{F}_{s,t}^x = \sigma\{x_{s,t}\}$ . Moreover, we define the  $D([0, t-s]; F)$ -valued random variable  $y_{s,t}$  and corresponding  $\sigma$ -fields

$$y_{s,t} = (y_r - y_s)_{r \in [s,t]}, \quad \mathcal{F}_{s,t}^y = \sigma\{y_{s,t}\}, \quad \mathcal{F}_{s,t} = \mathcal{F}_{s,t}^x \vee \mathcal{F}_{s,t}^y.$$

The shift  $\Theta^t : \Omega \rightarrow \Omega$  is defined as  $\Theta^t(\xi, \eta)(s) = (\xi(s+t), \eta(s+t) - \eta(t))$ . Let us emphasize that the observation segment  $y_{s,t}$  and the shift  $\Theta^t$  are defined differently than in the discrete time setting: the present choice accounts for the additivity of the observations, which we introduce next.

In the continuous time setting, we will assume that the canonical process is a hidden Markov process or *Markov additive process*: that is,  $(x_t, y_t)_{t \geq 0}$  is a time-homogeneous Markov process such that  $\mathbf{E}[f(x_t, y_t - y_0) | x_0, y_0]$  does not depend on  $y_0$  for any bounded measurable function  $f$ . It is not difficult to verify that this assumption corresponds to the following two properties: the process  $(x_t)_{t \geq 0}$  is Markov in its own right, and the process  $(y_t)_{t \geq 0}$  has conditionally independent increments given  $(x_t)_{t \geq 0}$  (see, for example, [6]).

In the following, we define the probability  $\mathbf{P}^x$  on  $\mathcal{F}_{0,\infty}$  as the law of the Markov additive process  $(x_t, y_t - y_0)_{t \geq 0}$  started at  $x_0 = x \in E$ , and let  $\mathbf{P}^\mu = \int \mathbf{P}^x \mu(dx)$  for  $\mu \in \mathcal{P}(E)$ . We will assume the existence of an invariant probability  $\lambda \in \mathcal{P}(E)$  so that  $\mathbf{P}^\lambda$  is invariant under the shift  $\Theta^t$  for all  $t \geq 0$ . We define  $\mathbf{P} = \mathbf{P}^\lambda$ , and introduce the continuous-time nonlinear filters

$$\pi_t^\mu = \mathbf{P}^\mu[x_t \in \cdot | \mathcal{F}_{0,t}^y], \quad \pi_t = \mathbf{P}[x_t \in \cdot | \mathcal{F}_{0,t}^y].$$

As we will consider convergence in probability only, we will not worry about the regularity of  $\pi_t^\mu$  as a function of  $t$  (that is, for each  $t \geq 0$ , we may choose any version of the above regular conditional probabilities).

The Markov additive process  $(x_t, y_t)_{t \geq 0}$  is said to be *nondegenerate* if for every  $\delta \in ]0, \infty[$ , there exists a reference measure  $\varphi_\delta$  on  $D([0, \delta]; F)$  and a strictly positive function  $g_\delta : D([0, \delta]; E \times F) \rightarrow ]0, \infty[$  such that

$$\mathbf{P}^z[y_{t,t+\delta} \in A | \mathcal{F}_{0,\infty}^x] = \int \mathbf{1}_A(\eta) g_\delta(x_{t,t+\delta}, \eta) \varphi_\delta(d\eta) \quad \mathbf{P}^z\text{-a.s.}$$

for all  $t \geq 0$ ,  $A \in \mathcal{B}(D([0, \delta]; F))$  and  $z \in E$ . This assumption is the direct counterpart of nondegeneracy for discrete time hidden Markov models.



4.4.2. *Local mixing in continuous time.* The aim of this section is to obtain a continuous-time version of Corollary 4.12 (in the setting of hidden Markov processes). To this end, we assume that the state space  $E$  of the unobserved process  $x_t$  is contained in a countable product  $E \subseteq \prod_{i \in I} E^i$ , where each  $E^i$  is Polish. Let  $x_t^J$  be the projection of  $x_t$  on  $\prod_{i \in J} E^i$  and

$$\mathcal{F}_{s,t}^J = \sigma\{x_{s,t}^J, y_{s,t}\} \quad \text{for } J \subseteq I, \quad s \leq t.$$

Let  $\mathcal{E}^J \subseteq \mathcal{B}(E)$  be the cylinder  $\sigma$ -field generated by the coordinates  $J \subseteq I$ .

**THEOREM 4.20** (Continuous local mixing filter stability). *If the Markov additive process  $(x_t, y_t)_{t \geq 0}$  is nondegenerate and locally mixing in the sense*

$$\|\mathbf{P}^x - \mathbf{P}\|_{\mathcal{F}_{t,\infty}^J} \xrightarrow{t \rightarrow \infty} 0 \quad \text{for all } x \in E \text{ and } J \subseteq I, \quad |J| < \infty,$$

*then the filter is stable in the sense that*

$$\|\pi_t^\mu - \pi_t^\nu\|_{\mathcal{E}^J} \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^\gamma\text{-probability} \quad \text{for all } J \subseteq I, \quad |J| < \infty$$

*for every  $\mu, \nu, \gamma \in \mathcal{P}(E)$ .*

We will reduce the proof to the discrete time case. The key to this reduction is the following lemma, essentially due to Blackwell and Dubins [2].

**LEMMA 4.21.** *Let  $\mathbf{R}$  and  $\mathbf{R}'$  be probabilities on  $D(\mathbb{R}_+, F)$ . Let  $r_t$  the coordinate process of  $D(\mathbb{R}_+, F)$  and  $\mathcal{G}_t = \sigma\{r_s : s \in [0, t]\}$ . If  $\mathbf{R} \ll \mathbf{R}'$ , then*

$$\|\mathbf{R}[\cdot | \mathcal{G}_t] - \mathbf{R}'[\cdot | \mathcal{G}_t]\| \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{R}\text{-probability.}$$

**PROOF.** Let  $\Lambda = d\mathbf{R}/d\mathbf{R}'$ . Then the Bayes formula yields

$$\mathbf{R}[A | \mathcal{G}_t] - \mathbf{R}'[A | \mathcal{G}_t] = \mathbf{E}_{\mathbf{R}'} \left[ \mathbf{1}_A \frac{\Lambda - \mathbf{E}_{\mathbf{R}'}[\Lambda | \mathcal{G}_t]}{\mathbf{E}_{\mathbf{R}'}[\Lambda | \mathcal{G}_t]} \middle| \mathcal{G}_t \right] \quad \mathbf{R}\text{-a.s.}$$

As  $D(\mathbb{R}_+, F)$  is countably generated, it follows that

$$\|\mathbf{R}[\cdot | \mathcal{G}_t] - \mathbf{R}'[\cdot | \mathcal{G}_t]\| \leq \frac{\mathbf{E}_{\mathbf{R}'}[\Delta_t | \mathcal{G}_t]}{\mathbf{E}_{\mathbf{R}'}[\Lambda | \mathcal{G}_t]} \quad \mathbf{R}\text{-a.s.}, \quad \Delta_t = |\Lambda - \mathbf{E}_{\mathbf{R}'}[\Lambda | \mathcal{G}_t]|.$$

But note that  $\mathbf{E}_{\mathbf{R}'}[\Lambda | \mathcal{G}_t] \rightarrow \Lambda$  and therefore  $\Delta_t \rightarrow 0$  in  $\mathbf{R}'$ -probability by the martingale convergence theorem (a right-continuous modification of the martingale is not needed for convergence in probability), while  $\Lambda > 0$   $\mathbf{R}$ -a.s. The remaining steps of the proof follow the proof of Lemma 4.9.  $\square$

We now turn to the proof of Theorem 4.20.

PROOF OF THEOREM 4.20. Let  $\bar{E} = D([0, 1]; E)$ ,  $\bar{F} = D([0, 1]; F)$ ,  $X_n = x_{n,n+1}$ , and  $Y_n = y_{n,n+1}$ . Then  $(X_n, Y_n)_{n \geq 0}$  is a nondegenerate hidden Markov model in  $\bar{E} \times \bar{F}$  under  $\mathbf{P}^\mu$  for every  $\mu \in \mathcal{P}(E)$ : in particular,  $(X_n)_{n \geq 0}$  is a Markov chain with initial measure  $\bar{\mu}$  and transition kernel  $P_0$  given by

$$\bar{\mu}(d\xi) = \mathbf{P}^\mu[x_{0,1} \in d\xi], \quad P_0(\xi, d\xi') = \mathbf{P}^{\xi(1)}[x_{0,1} \in d\xi'],$$

while, by the nondegeneracy assumption, the observation kernel  $\Phi$  is

$$\Phi(\xi, d\eta) = g_1(\xi, \eta) \varphi_1(d\eta)$$

(so that, as in section 4.3.2,  $(X_n, Y_n)_{n \geq 0}$  is the Markov chain with transition kernel  $P(\xi, \eta, d\xi', d\eta') = P_0(\xi, d\xi')\Phi(\xi', d\eta')$ ). Moreover,  $\bar{\lambda} = \mathbf{P}[x_{0,1} \in \cdot]$  is an invariant probability for the discrete-time model  $(X_n, Y_n)_{n \geq 0}$ .

We can now apply Corollary 4.12 to the discrete-time model  $(X_n, Y_n)_{n \geq 0}$ . Indeed, we can decompose  $\bar{E} \subseteq \prod_{i \in I} \bar{E}^i$  with  $\bar{E}^i = D([0, 1]; E^i)$ , and our local mixing assumption directly implies that the discrete model  $(X_n, Y_n)_{n \geq 0}$  is locally mixing with respect to this decomposition. It follows that

$$\|\mathbf{P}^\mu[x_{n,n+1}^J \in \cdot | \mathcal{F}_{0,n+1}^y] - \mathbf{P}[x_{n,n+1}^J \in \cdot | \mathcal{F}_{0,n+1}^y]\| \xrightarrow{n \rightarrow \infty} 0 \quad \mathbf{P}\text{-a.s.}$$

for all  $J \subseteq I$ ,  $|J| < \infty$  and  $\mu \in \mathcal{P}(E)$  by Corollary 4.12, while Corollary 4.11 yields the equivalence  $\mathbf{P}^\mu|_{\mathcal{F}_{0,\infty}^y} \sim \mathbf{P}|_{\mathcal{F}_{0,\infty}^y}$ . The latter implies that

$$\|\mathbf{P}^\mu[y_{0,\infty} \in \cdot | \mathcal{F}_{0,t}^y] - \mathbf{P}[y_{0,\infty} \in \cdot | \mathcal{F}_{0,t}^y]\| \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}\text{-probability}$$

by Lemma 4.21, which we now proceed to exploit.

Let  $t \in [n, n+1]$  for some  $n \in \mathbb{N}$ . Then we have

$$\pi_t^\mu = \mathbf{E}^\mu[\mathbf{P}^\mu[x_t \in \cdot | \mathcal{F}_{0,n+1}^y] | \mathcal{F}_{0,t}^y], \quad \pi_t = \mathbf{E}[\mathbf{P}[x_t \in \cdot | \mathcal{F}_{0,n+1}^y] | \mathcal{F}_{0,t}^y].$$

We can therefore estimate

$$\begin{aligned} \|\pi_t^\mu - \pi_t\|_{\mathcal{E}^J} &\leq \|\mathbf{P}^\mu[y_{0,\infty} \in \cdot | \mathcal{F}_{0,t}^y] - \mathbf{P}[y_{0,\infty} \in \cdot | \mathcal{F}_{0,t}^y]\| \\ &\quad + \mathbf{E}[\|\mathbf{P}^\mu[x_t \in \cdot | \mathcal{F}_{0,n+1}^y] - \mathbf{P}[x_t \in \cdot | \mathcal{F}_{0,n+1}^y]\|_{\mathcal{E}^J} | \mathcal{F}_{0,t}^y]. \end{aligned}$$

It follows that  $\|\pi_t^\mu - \pi_t\|_{\mathcal{E}^J} \rightarrow 0$  as  $t \rightarrow \infty$  in  $\mathbf{P}$ -probability. As  $\mu$  was arbitrary, the proof is easily completed using the triangle inequality and the equivalence of all observation laws to  $\mathbf{P}|_{\mathcal{F}_{0,\infty}^y}$  as established above.  $\square$

REMARK 4.22. As we have seen above, deducing filter stability in continuous time from our discrete time results requires some additional arguments (a slightly longer argument will be used below in the setting of asymptotic coupling). Let us therefore note, for sake of completeness, that the corresponding results on the ergodicity of the filtering process  $(\pi_t)_{t \geq 0}$  as in section 4.3.3 follow immediately from their discrete-time counterparts: in fact, uniqueness of the invariant measure of any continuous-time Markov process  $(\pi_t)_{t \geq 0}$  is evidently implied by uniqueness for the discretely sampled process  $(\pi_n)_{n \in \mathbb{N}}$ . There is therefore no need to consider this question separately.

4.4.3. *Asymptotic coupling in continuous time.* We now turn to the problem of obtaining a continuous-time counterpart to our asymptotic coupling filter stability Theorem 4.13. To this end we will assume, as we have done throughout this section, that  $(x_t, y_t)_{t \geq 0}$  is a Markov additive process in the Polish state space  $E \times F$ . In addition, we will assume in this subsection that the unobserved process  $(x_t)_{t \geq 0}$  has continuous sample paths. While this is not absolutely essential, the restriction to continuous processes facilitates the treatment of asymptotic couplings in continuous time.

The following is the main result of this section. As in Theorem 4.13, we will fix in the following a complete metric  $d$  for the Polish space  $E$ .

THEOREM 4.23 (Continuous weak-\* filter stability). *Let  $(x_t, y_t)_{t \geq 0}$  be a nondegenerate Markov additive process that admits an invariant probability  $\lambda$ , and assume that the unobserved process  $(x_t)_{t \geq 0}$  has continuous sample paths. Moreover, let  $\tilde{d}(x, y) \geq d(x, y)$  for all  $x, y \in E$ , fix  $\Delta > 0$ , and define the intervals  $I_n = [n\Delta, (n+1)\Delta]$ . Suppose that the following hold:*

a. *There exists  $\alpha > 0$  such that*

$$\forall x, x' \in E \quad \exists \mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'}) \quad \text{s.t.} \quad \mathbf{Q} \left[ \sum_{n=1}^{\infty} \sup_{t \in I_n} \tilde{d}(x_t, x'_t)^2 < \infty \right] \geq \alpha.$$

b. *There exists  $C < \infty$  such that for all  $\delta \leq \Delta$  and  $\xi, \xi' \in C([0, \delta]; E)$*

$$\int \{ \sqrt{g_\delta(\xi, \eta)} - \sqrt{g_\delta(\xi', \eta)} \}^2 \varphi_\delta(d\eta) \leq C \sup_{t \in [0, \delta]} \tilde{d}(\xi(t), \xi'(t))^2.$$

*Then the filter is stable in the sense that*

$$\|\pi_t^\mu - \pi_t^\nu\|_{\text{BL}} \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^\gamma\text{-probability for all } \mu, \nu, \gamma \in \mathcal{P}(E).$$

PROOF. We begin by noting that if assumption  $a$  holds for  $\Delta$ , then this assumption also holds if  $\Delta$  is replaced by  $\Delta/r$  for some  $r \in \mathbb{N}$ . Indeed, as

$$\sum_{n=1}^{\infty} \sup_{t \in I_n} \tilde{d}(x_t, x'_t)^2 \geq \frac{1}{r} \sum_{k=1}^r \sum_{n=1}^{\infty} \sup_{t \in [(n+(k-1)/r)\Delta, (n+k/r)\Delta]} \tilde{d}(x_t, x'_t)^2,$$

the claim follows. Fix  $r \in \mathbb{N}$  for the time being. Define  $\bar{E} = C([0, \Delta/r]; E)$ ,  $\bar{F} = D([0, \Delta/r]; F)$ ,  $X_n = x_{t_n, t_{n+1}}$ , and  $Y_n = y_{t_n, t_{n+1}}$ , where  $t_n = n\Delta/r$ . Then it follows as in the proof of Theorem 4.20 that  $(X_n, Y_n)_{n \geq 0}$  is a nondegenerate hidden Markov model in  $\bar{E} \times \bar{F}$  that admits an invariant probability (note that the definition of  $\bar{E}$  takes into account that  $(x_t)_{t \geq 0}$  has continuous sample paths). Moreover, if we endow  $\bar{E}$  with the metric  $\bar{d}(\xi, \xi') = \sup_{t \in [0, \Delta/r]} d(\xi(t), \xi'(t))$ , then evidently the assumptions of Theorem 4.13 are satisfied. It follows that for any  $f \in \text{Lip}(\bar{E})$  and  $\mu \in \mathcal{P}(E)$

$$|\mathbf{E}^\mu[f(x_{t_{n-1}, t_n}) | \mathcal{F}_{0, t_n}^y] - \mathbf{E}[f(x_{t_{n-1}, t_n}) | \mathcal{F}_{0, t_n}^y]| \xrightarrow{n \rightarrow \infty} 0$$

$\mathbf{P}$ -a.s. To proceed, let us fix  $g \in \text{Lip}(E)$ , and define  $G(\xi_{0, \Delta/r}) = g(\xi(0))$  and  $\bar{G}(\xi_{0, \Delta/r}) = \sup_{s \in [0, \Delta/r]} |g(\xi(0)) - g(\xi(s))|$ . We can easily estimate

$$\begin{aligned} |\mathbf{E}^\mu[g(x_t) | \mathcal{F}_{0, t_n}^y] - \mathbf{E}[g(x_t) | \mathcal{F}_{0, t_n}^y]| &\leq |\mathbf{E}^\mu[g(x_{t_{n-1}}) | \mathcal{F}_{0, t_n}^y] - \mathbf{E}[g(x_{t_{n-1}}) | \mathcal{F}_{0, t_n}^y]| \\ &\quad + |\mathbf{E}^\mu[\bar{G}(x_{t_{n-1}, t_n}) | \mathcal{F}_{0, t_n}^y] - \mathbf{E}[\bar{G}(x_{t_{n-1}, t_n}) | \mathcal{F}_{0, t_n}^y]| + 2 \mathbf{E}[\bar{G}(x_{t_{n-1}, t_n}) | \mathcal{F}_{0, t_n}^y]| \end{aligned}$$

for any  $t \in [t_{n-1}, t_n]$ . As  $G$  and  $\bar{G}$  are  $\bar{d}$ -Lipschitz, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{t \in [t_{n-1}, t_n]} \mathbf{E}[|\mathbf{E}^\mu[g(x_t) | \mathcal{F}_{0, t_n}^y] - \mathbf{E}[g(x_t) | \mathcal{F}_{0, t_n}^y]|] \\ \leq 2 \mathbf{E} \left[ \sup_{s \in [0, \Delta/r]} |g(x_0) - g(x_s)| \right]. \end{aligned}$$

On the other hand, we can estimate as in the proof of Theorem 4.20

$$\begin{aligned} \mathbf{E}[|\pi_t^\mu(g) - \pi_t(g)|] &\leq \mathbf{E}[|\mathbf{P}^\mu[y_{0, \infty} \in \cdot | \mathcal{F}_{0, t}^y] - \mathbf{P}[y_{0, \infty} \in \cdot | \mathcal{F}_{0, t}^y]|] \\ &\quad + \mathbf{E}[|\mathbf{E}^\mu[g(x_t) | \mathcal{F}_{0, t_n}^y] - \mathbf{E}[g(x_t) | \mathcal{F}_{0, t_n}^y]|] \end{aligned}$$

for  $t \in [t_{n-1}, t_n]$ . Applying Lemma 4.21 as in Theorem 4.20 yields

$$\limsup_{t \rightarrow \infty} \mathbf{E}[|\pi_t^\mu(g) - \pi_t(g)|] \leq 2 \mathbf{E} \left[ \sup_{s \in [0, \Delta/r]} |g(x_0) - g(x_s)| \right].$$

But note that this holds for any  $r \in \mathbb{N}$ . Letting  $r \rightarrow \infty$ , we obtain

$$|\pi_t^\mu(g) - \pi_t(g)| \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}\text{-probability}$$

using the continuity of paths. Finally, note that  $g \in \text{Lip}(E)$  is arbitrary. We can therefore strengthen the convergence for individual  $g$  to  $\|\cdot\|_{\text{BL}}$ -convergence as in the proof of Theorem 4.13. The proof is now easily completed using the triangle inequality and the equivalence of all observation laws to  $\mathbf{P}|_{\mathcal{G}_{0,\infty}^y}$  as established in the proof of Theorem 4.20.  $\square$

**5. Examples.** Infinite-dimensional Markov processes arise in a diverse range of applications, see, for example, the monograph [9]. The aim of this section is to demonstrate that the theory that we have developed in the previous sections is directly applicable in several different settings. In section 5.1, we consider the simplest possible example of an infinite-dimensional system: a stochastic heat equation with smooth forcing and point observations. While this example is nearly trivial, it allows us to easily demonstrate our results in the simplest possible setting. In section 5.2, we consider a highly degenerate stochastic Navier-Stokes equation with Eulerian observations. In section 5.3 we consider stochastic spin systems. Finally, in section 5.4 we consider filtering problems for stochastic differential delay equations.

Our original motivation for developing the theory in this paper was to understand the nonlinear filtering problem for stochastic Navier-Stokes equations with Lagrangian observations of the position of a passive tracer, cf. [38, section 3.6] (such problems arise in oceanography; [38] also describes several other infinite-dimensional filtering problems that arise in applications). This example certainly fits in our general theory, the main difficulty being verification of the (unconditional) ergodicity properties of the passive tracer which have not been previously considered in the literature. In order not to unduly lengthen this paper, the details will be given elsewhere. As the aim of this section is to illustrate the wide applicability of our abstract theory, rather than the development of a specific application, we restrict attention to examples that are readily studied using existing ergodicity results.

**5.1. Stochastic heat equation.** We investigate the following example from [28]. Consider the stochastic heat equation on the unit interval  $x \in [0, 1]$ :

$$du(t, z) = \Delta u(t, z) dt + dw(t, z), \quad u(t, 0) = u(t, 1) = 0.$$

Here  $dw(t, z)$  is the white in time, smooth in space random forcing

$$w(t, z) = \sum_{k=1}^{\infty} \sigma_k \sqrt{2} \sin(\pi k z) W_t^k,$$

$(W_t^k)_{t \geq 0}$ ,  $k \in \mathbb{N}$  are independent Brownian motions,  $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$ , and  $\sigma_k > 0$  for all  $k \in \mathbb{N}$ . We will assume that  $u(t, z)$  is observed at the points

$z_1, \dots, z_n \in [0, 1]$  and that the observations are corrupted by independent white noise: that is, we introduce the  $\mathbb{R}^n$ -valued observation model

$$dy_t^i = u(t, z_i) dt + dB_t^i, \quad i = 1, \dots, n,$$

where  $(B_t^i)_{t \geq 0}$ ,  $i = 1, \dots, n$  are independent Brownian motions that are independent of  $(W_t^k)_{t \geq 0}$ ,  $k \in \mathbb{N}$ . As we are working with Dirichlet boundary conditions, we view  $z \mapsto u(t, z)$  as taking values in the Hilbert subspace  $H \subset L^2[0, 1]$  spanned by the eigenfunctions  $(e_k)_{k \in \mathbb{N}}$ ,  $e_k(z) = \sqrt{2} \sin(\pi k z)$ .

LEMMA 5.1. *Let  $x_t = u(t, \cdot)$ . Then the pair  $(x_t, y_t)_{t \geq 0}$  defines a nondegenerate Markov additive process in  $H \times \mathbb{R}^n$  with continuous paths. Moreover, the unobserved process  $(x_t)_{t \geq 0}$  admits a unique invariant probability  $\lambda$ .*

PROOF. It is easily seen that for any  $u(0, \cdot) \in H$ , the equation for  $u(t, \cdot)$  has a unique mild solution in  $H$  that has continuous paths and satisfies the Markov property (cf. [9]). If we expand  $x_t = \sum_{k=1}^{\infty} x_t^k e_k$ , then evidently

$$dx_t^k = -\pi^2 k^2 x_t^k dt + \sigma_k dW_t^k.$$

By Itô's formula, we obtain

$$\mathbf{E}[\|x_t\|_H^2] + \mathbf{E}\left[\int_0^t 2\|x_s\|_{H^1}^2 ds\right] = \|x_0\|_H^2 + \|\sigma\|_H^2 t,$$

where we defined the Sobolev norm  $\|x_t\|_{H^s}^2 = \sum_{k=1}^{\infty} (\pi k)^{2s} (x_t^k)^2$ . Note that  $|u(t, z)| \leq \sqrt{2} \sum_{k=1}^{\infty} |x_t^k| \leq 3^{-1/2} \|x_t\|_{H^1}$  by Cauchy-Schwarz. Thus  $z \mapsto u(t, z)$  is continuous for a.e.  $t$ , so the observation process  $y_t$  is well defined and the pair  $(x_t, y_t)_{t \geq 0}$  defines a Markov additive process. Moreover,

$$\mathbf{E}\left[\int_0^t |u(s, z_i)|^2 ds\right] \leq \frac{\|x_0\|_H^2 + \|\sigma\|_H^2 t}{6} < \infty.$$

Therefore, by Girsanov's theorem, the conditional law of  $y_{0,t}$  given  $x_{0,t}$  is equivalent to the Wiener measure a.s. for any  $t < \infty$  and  $x_0 \in H$  (as  $(u(t, z_i))_{t \geq 0}$  and  $(B_t^i)_{t \geq 0}$  are independent, Novikov's criterion can be applied conditionally). This establishes the nondegeneracy assumption. Finally, as each Fourier mode  $x_t^k$  is an independent Ornstein-Uhlenbeck process, it is easily seen by explicit computation that the law of  $x_t$  converges weakly as  $t \rightarrow \infty$  to a unique Gaussian product measure  $\lambda$  for any  $x_0 \in H$ .  $\square$

It is evident from Lemma 5.1 that the ergodic theory of  $u(t, z)$  is quite trivial: each of the Fourier modes is an independent ergodic one-dimensional

Ornstein-Uhlenbeck process (recall Example 2.3). Nonetheless, the reader may easily verify using the Kakutani theorem [36, p. 531] that  $(x_t)_{t \geq 0}$  is not Harris when the forcing is sufficiently smooth (for example,  $\sigma_k = e^{-k^3}$ ). Moreover, the finite-dimensional projections  $(x_t^1, \dots, x_t^k, y_t)$  are not Markovian. Thus stability of the corresponding nonlinear filter does not follow from earlier results. While this example remains essentially trivial, it is nonetheless instructive to illustrate our results in this simplest possible setting.

5.1.1. *Local mixing.* To every  $x = \sum_{k=1}^{\infty} x_k e_k \in H$  we identify a vector of Fourier coefficients  $(x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ . In order to apply our local mixing results, we can therefore view  $H$  as a subset of the product space  $\mathbb{R}^{\mathbb{N}}$ . Note that  $H$  is certainly not a topological subspace of  $\mathbb{R}^{\mathbb{N}}$  (pointwise convergence of the Fourier coefficients does not imply convergence in  $H$ ); however,  $H$  is a measurable subspace of  $\mathbb{R}^{\mathbb{N}}$ , which is all that is needed in the present setting.

For every  $k \in \mathbb{N}$ , define the local  $\sigma$ -fields

$$\mathcal{F}_{s,t}^k = \sigma\{x_{s,t}^1, \dots, x_{s,t}^k, y_{s,t}\}, \quad s \leq t.$$

To apply Theorem 4.20, it suffices to establish the local mixing property.

LEMMA 5.2. *The Markov additive process  $(x_t, y_t)_{t \geq 0}$  is locally mixing:*

$$\|\mathbf{P}^x - \mathbf{P}\|_{\mathcal{F}_{t,\infty}^k} \xrightarrow{t \rightarrow \infty} 0 \quad \text{for every } x \in H, k \in \mathbb{N}.$$

PROOF. Let  $x, x' \in H$ , and define  $v \in H$  such that  $\langle e_\ell, v \rangle = \langle e_\ell, x \rangle$  for  $1 \leq \ell \leq k$ , and  $\langle e_\ell, v \rangle = \langle e_\ell, x' \rangle$  for  $\ell > k$ . It is easily seen that

$$\|\mathbf{P}^x - \mathbf{P}^{x'}\|_{\mathcal{F}_{t,\infty}^k} \leq \|\mathbf{P}^x - \mathbf{P}^v\|_{\mathcal{F}_{t,\infty}^k} + \|\mathbf{P}^{x'} - \mathbf{P}^v\|_{\sigma\{x_t\}}$$

by the Markov additive property. As the Fourier modes are independent, we evidently have  $\|\mathbf{P}^{x'} - \mathbf{P}^v\|_{\sigma\{x_t\}} = \|\mathbf{P}^{x'} - \mathbf{P}^v\|_{\sigma\{x_t^1, \dots, x_t^k\}} \rightarrow 0$  as  $t \rightarrow \infty$  (for example, by explicit computation of the law of the  $k$ -dimensional Ornstein-Uhlenbeck process). It therefore remains to consider the first term.

Construct on a larger probability space  $(\Omega', \mathcal{F}', \mathbf{Q})$  the triple  $(x_t, v_t, y_t)_{t \geq 0}$  as follows. The processes  $x_t$  and  $v_t$  are solutions to the stochastic heat equation driven by the same Brownian motion realization, but with different initial conditions  $x_0 = x$  and  $v_0 = v$ , while  $dy_t^i = x_t(z_i) dt + dB_t^i$  as above. Now note that can show precisely as in the proof of Lemma 5.1 that

$$\mathbf{E} \left[ \int_0^\infty |v_s(z_i) - x_s(z_i)|^2 ds \right] \leq \mathbf{E} \left[ \frac{1}{3} \int_0^\infty \|v_s - x_s\|_{H^1}^2 ds \right] \leq \frac{\|x - v\|_H^2}{6} < \infty.$$

As  $(x_t, v_t)_{t \geq 0}$  is independent of  $(B_t)_{t \geq 0}$ , we can apply Novikov's criterion conditionally to establish that  $\mathbf{E}[\Lambda_t] = 1$  for any  $t \geq 0$ , where we define

$$\Lambda_t = \prod_{i=1}^n \exp \left[ \int_t^\infty \{v_s(z_i) - x_s(z_i)\} dB_s^i - \frac{1}{2} \int_t^\infty |v_s(z_i) - x_s(z_i)|^2 ds \right].$$

Using Girsanov's theorem, we obtain for any  $A \in \mathcal{F}_{t,\infty}^k$

$$\mathbf{P}^z(A) = \mathbf{E}_{\mathbf{Q}}[\mathbf{1}_A(v_{t,\infty}^1, \dots, v_{t,\infty}^k, y_{t,\infty}) \Lambda_t] = \mathbf{E}_{\mathbf{Q}}[\mathbf{1}_A(x_{t,\infty}^1, \dots, x_{t,\infty}^k, y_{t,\infty}) \Lambda_t],$$

where we have used that  $x_t^\ell = v_t^\ell$  for all  $t \geq 0$  when  $\ell \leq k$ . Moreover, the law of  $(x_t, y_t)_{t \geq 0}$  under  $\mathbf{Q}$  obviously coincides with  $\mathbf{P}^x$ . We therefore conclude that  $\|\mathbf{P}^x - \mathbf{P}^z\|_{\mathcal{F}_{t,\infty}^k} \leq \mathbf{E}_{\mathbf{Q}}[|\Lambda_t - 1|] \rightarrow 0$  as  $t \rightarrow \infty$  by Scheffé's lemma.  $\square$

Let  $\mathcal{E}^k = \sigma\{P_k\}$ , where  $P_k : H \rightarrow H$  is the projection onto the first  $k$  Fourier modes. Theorem 4.20 immediately yields the filter stability result

$$\|\pi_t^\mu - \pi_t^\nu\|_{\mathcal{E}^k} \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^\gamma\text{-probability} \quad \text{for all } k \in \mathbb{N}, \mu, \nu, \gamma \in \mathcal{P}(H).$$

A simple tightness argument can be used to deduce also filter stability in the bounded-Lipschitz norm from this statement. However, let us demonstrate instead how the latter can be obtained directly from Theorem 4.23.

**5.1.2. Asymptotic coupling.** It was shown in Lemma 5.1 that  $(x_t, y_t)_{t \geq 0}$  is nondegenerate. It follows from the proof that we may choose the reference measure  $\varphi_\delta$  to be the Wiener measure on  $D([0, \delta]; \mathbb{R}^n)$  and that

$$g_\delta(\xi, \eta) = \prod_{i=1}^n \exp \left[ \int_0^\delta \xi(s, z_i) d\eta^i(s) - \frac{1}{2} \int_0^\delta |\xi(s, z_i)|^2 ds \right]$$

for  $\xi \in C([0, \delta]; H) \cap L^2([0, \delta], H^1)$  (for simplicity, let  $g_\delta(\xi, \eta) = 1$  otherwise). We begin by establishing the Lipschitz property of the observations.

**LEMMA 5.3.** *For all  $\delta \leq 1$  and  $\xi, \xi' \in C([0, \delta]; H)$*

$$\int \{ \sqrt{g_\delta(\xi, \eta)} - \sqrt{g_\delta(\xi', \eta)} \}^2 \varphi_\delta(d\eta) \leq \frac{n}{12} \sup_{t \in [0, \delta]} \|\xi(t) - \xi'(t)\|_{H^1}^2.$$

**PROOF.** The result is trivial unless  $\xi, \xi' \in L^2([0, \delta]; H^1)$ , in which case

$$\int \{ \sqrt{g_\delta(\xi, \eta)} - \sqrt{g_\delta(\xi', \eta)} \}^2 \varphi_\delta(d\eta) = 2 - 2e^{-\frac{1}{8} \int_0^\delta \sum_{i=1}^n |\xi(s, z_i) - \xi'(s, z_i)|^2 ds}.$$

Now use  $1 - e^{-x} \leq x$  and  $|\xi(s, z_i) - \xi'(s, z_i)| \leq 3^{-1/2} \|\xi(s) - \xi'(s)\|_{H^1}$ .  $\square$



Thus the second assumption of Theorem 4.23 is satisfied for  $\tilde{d}(x, y) = \|x - y\|_{H^1}$  and  $\Delta = 1$ . It is clear that the observations cannot be continuous with respect to  $\|\cdot\|_H$ , which is the reason that we have introduced the pseudodistance  $\tilde{d}$  in Theorem 4.23. To establish filter stability, it remains to produce an asymptotic coupling in  $H^1$ , which is trivial in this example.

LEMMA 5.4. *For all  $x, x' \in H$ , there exists  $\mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'})$  such that*

$$\sum_{n=1}^{\infty} \sup_{t \in [n, n+1]} \|x_t - x'_t\|_{H^1}^2 < \infty \quad \mathbf{Q}\text{-a.s.}$$

PROOF. Choose  $\mathbf{Q}$  such that the processes  $x_t$  and  $x'_t$  are solutions to the stochastic heat equation driven by the same Brownian motion realization, but with different initial conditions  $x_0 = x$  and  $x'_0 = x'$ . Then

$$\rho_t = x_t - x'_t, \quad d\rho_t^k = -\pi^2 k^2 \rho_t^k dt$$

as in the proof of Lemma 5.1. As the difference  $\rho_t$  is deterministic, the result follows readily (for example,  $\|\rho_t\|_{H^1}$  can be computed explicitly).  $\square$

As we have verified all the assumptions of Theorem 4.23, it follows that

$$\|\pi_t^\mu - \pi_t^\nu\|_{\text{BL}} \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^\gamma\text{-probability for all } \mu, \nu, \gamma \in \mathcal{P}(E),$$

that is, we have established filter stability in the bounded-Lipschitz norm.

REMARK 5.5. Beside that it admits a trivial ergodic theory, the example considered this section is special in that it is a linear Gaussian model. In finite dimension, such filtering problems are amenable to explicit analysis as the filter reduces to the well-known Kalman filter, which is a rather simple linear equation [25]. Some results in this direction for linear stochastic evolution equations were considered by Vinter [44]. However, the present example does not fit in the setting of [44] as the observation operator  $C : H \rightarrow \mathbb{R}^n$ ,  $Cu = (u(z_1), \dots, u(z_n))$  is unbounded, which significantly complicates even the definition of the Kalman filtering equations in infinite dimension. It is therefore interesting to note the ease with which we have obtained stability results from our general nonlinear theory even in this trivial linear example.

5.2. *Stochastic Navier-Stokes equation.* We now turn to a much less trivial example inspired by [38, section 3.6]: we will consider discrete time Eulerian (point) observations of the velocity of a fluid that is modeled by a Navier-Stokes equation with white in time, smooth in space random forcing.

We consider a velocity field  $u(t, z) \in \mathbb{R}^2$  on the two-dimensional torus  $z \in \mathbb{T}^2 = [-\pi, \pi]^2$  such that  $\int u(t, z) dz = 0$  and  $\nabla \cdot u(t, z) = 0$  for all  $t \geq 0$ . The dynamics of  $u(t, z)$  are given by the stochastic Navier-Stokes equation

$$du(t, z) = \{\nu \Delta u(t, z) - (u(t, z) \cdot \nabla)u(t, z) - \nabla p(t, z)\} dt + d\tilde{w}(t, z)$$

with periodic boundary conditions, where  $\nu > 0$  is the fluid viscosity,  $\tilde{w}$  is a spatial mean zero stochastic forcing to be specified later, and the pressure  $p$  is chosen to enforce the divergence-free constraint  $\nabla \cdot u(t, z) = 0$ .

To define the observations, let us fix points  $z_1, \dots, z_r \in \mathbb{T}^2$  at which the fluid velocity is measured. We assume that measurements are taken at the discrete time instants  $t_n = n\delta$ ,  $n \geq 0$ , where we fix the sampling interval  $\delta > 0$  throughout this section. The observations are then given by<sup>1</sup>

$$Y_n^i = u(t_n, z_i) + \xi_n^i, \quad i = 1, \dots, r, \quad n \geq 0,$$

where  $(\xi_n)_{n \geq 0}$  are i.i.d.  $\mathbb{R}^{2r}$ -dimensional Gaussian random variables with nondegenerate covariance that are independent of  $(u(t, \cdot))_{t \geq 0}$ .

Following [16], it will be convenient to eliminate the divergence-free constraint from the stochastic Navier-Stokes equation by passing to an equivalent formulation. Define the vorticity  $v(t, z) = \nabla \times u(t, z) = \partial u^1(t, z)/\partial z^2 - \partial u^2(t, z)/\partial z^1$ , which is a scalar field on  $\mathbb{T}^2$ . As  $u$  is divergence-free and has spatial mean zero, we can reconstruct the velocity field from the vorticity as  $u = \mathcal{K}v$ , where the integral operator  $\mathcal{K}$  is defined in the Fourier domain as  $\langle e_k, \mathcal{K}v \rangle = -i(k^\perp/|k|^2)\langle e_k, v \rangle$  with  $e_k(z) = (2\pi)^{-1}e^{ik \cdot z}$ ,  $k \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ , and  $k^\perp = (k^2, -k^1)$ . In terms of vorticity, the Navier-Stokes equation reads

$$dv(t, z) = \{\nu \Delta v(t, z) - \mathcal{K}v(t, z) \cdot \nabla v(t, z)\} dt + dw(t, z),$$

where  $w(t, z) = \nabla \times \tilde{w}(t, z)$ , and the observation equation becomes

$$Y_n^i = \mathcal{K}v(t_n, z_i) + \xi_n^i, \quad i = 1, \dots, r, \quad n \geq 0.$$

From now on we will work with the vorticity equation, which we consider as an evolution equation in the Hilbert space  $H = \{v \in L^2(\mathbb{T}^2) : \int v(z) dz = 0\}$ . This formulation is equivalent to considering the original stochastic Navier-Stokes equation in  $\{u \in H^1 : \nabla \cdot u = 0, \int u(z) dz = 0\}$ . We also define the Sobolev norm  $\|v\|_{H^s}^2 = \sum_k |k|^{2s} |\langle e_k, v \rangle|^2$  and  $H^s = \{v \in H : \|v\|_{H^s} < \infty\}$ .

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<sup>1</sup> The observation equation makes sense when  $u(t, \cdot) \in H^2$ , as this implies that  $z \mapsto u(t, z)$  is continuous by the Sobolev embedding theorem. For concreteness, we can define  $Y_n^i = u(t_n, z_i) \mathbf{1}_{u(t_n, \cdot) \in H^2} + \xi_n^i$  which makes sense for any velocity field. As we will always work under assumptions that ensure sufficient smoothness of the solutions of the stochastic Navier-Stokes equations for all  $t > 0$ , this minor point will not affect our results.

It remains to specify the structure of the forcing  $w(t, z)$ . As in [16], we let  $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{(0, 0)\} = \mathbb{Z}_+^2 \cup \mathbb{Z}_-^2$  with  $\mathbb{Z}_+^2 = \{k \in \mathbb{Z}^2 : k^2 > 0 \text{ or } k^2 = 0, k^1 > 0\}$  and  $\mathbb{Z}_-^2 = -\mathbb{Z}_+^2$ , and we define the trigonometric basis  $f_k(z) = \sin(k \cdot z)$  for  $k \in \mathbb{Z}_+^2$  and  $f_k(z) = \cos(k \cdot z)$  for  $k \in \mathbb{Z}_-^2$ . The forcing is now given by

$$w(t, z) = \sum_{k \in \mathbb{Z}_0^2} \sigma_k f_k(z) W_t^k,$$

where  $(W_t^k)_{t \geq 0}$ ,  $k \in \mathbb{Z}_0^2$  are independent standard Brownian motions, and we will assume that  $\sum_k |k|^2 \sigma_k^2 < \infty$  (so that the forcing is in  $H^1$ ).

LEMMA 5.6. *Let  $X_n = v(t_n, \cdot)$ . Then  $(X_n, Y_n)_{n \geq 0}$  defines a nondegenerate hidden Markov model in  $H \times \mathbb{R}^{2r}$  that admits an invariant probability.*

PROOF. It is well known that the stochastic Navier-Stokes equation defines a stochastic flow, see [16, 20] and the references therein. Under our assumptions, this implies that the vorticity equation defines a Markov process in  $H$ . Thus  $(X_n, Y_n)_{n \geq 0}$  is evidently a hidden Markov model, and nondegeneracy follows as the observation kernel has a nondegenerate Gaussian density. Moreover, as we assumed that  $\sum_k |k|^2 \sigma_k^2 < \infty$ , standard Sobolev estimates (for example, [20, Proposition 2.4.12]) show that  $v(t, \cdot) \in H^1$  for all  $t > 0$  a.s. for any initial condition  $v(0, \cdot) \in H$ . Thus  $u(t, \cdot) = \mathcal{K}v(t, \cdot) \in H^2$  for all  $t > 0$  a.s., and the observation model is defined as intended. The existence of an invariant probability is standard (for example, [9, 20]).  $\square$

Our aim is now to establish stability of the nonlinear filter for the hidden Markov model  $(X_n, Y_n)_{n \geq 0}$ . This is much more difficult than for the heat equation in the previous section. First, in the present case the Fourier modes are coupled by the nonlinear term in the equation, so that energy can move across scales. Second, unlike in the heat equation example, only sufficiently fine scales are contracting. Nonetheless, in the case that all Fourier modes are forced (that is,  $\sigma_k > 0$  for all  $k \in \mathbb{Z}_0^2$ ), it is possible to establish local mixing using the Girsanov method developed in [13, 27]. In fact, the approach taken in these papers is well suited to our local zero-two laws (for example, Lemmas 3.1 and 3.2 in [13] can be used directly in conjunction with Corollary 2.8 to establish absolute regularity of a finite number of Fourier modes, and some additional effort yields the assumptions of Corollary 2.5). However, these methods do not extend to the degenerate setting.

We intend to illustrate that our results are applicable even in highly degenerate situations. To this end, we adopt the following assumptions [16].

ASSUMPTION 5.7. Let  $\mathcal{Z} = \{k \in \mathbb{Z}_0^2 : \sigma_k \neq 0\}$  be the set of forced modes. We assume that (a)  $\mathcal{Z}$  is a finite set; (b)  $\mathcal{Z} = -\mathcal{Z}$ ; (c) there exist  $k, k' \in \mathcal{Z}$  with  $|k| \neq |k'|$ ; (d) integer linear combinations of elements of  $\mathcal{Z}$  generate  $\mathbb{Z}^2$ .

It was shown by Hairer and Mattingly [16] that under these (essentially minimal) assumptions the stochastic Navier-Stokes equation is uniquely ergodic. In the remainder of this section, we will show that this assumption also ensures stability of the filter in the bounded-Lipschitz norm. Let us emphasize that no new ergodic theory is needed: we will simply verify the assumptions of Theorem 4.13 by a direct application of the machinery developed in [16, 17], together with a standard interpolation argument.

We will use the following tool to construct asymptotic couplings.

THEOREM 5.8. *Let  $Q$  be a transition kernel on  $H$ , and consider a continuous function  $W : H \rightarrow [1, \infty[$ . Suppose that for every  $\varphi \in C^1(H)$*

$$\|\nabla Q\varphi(x)\|_H \leq W(x) \left( C_1 \{Q\|\nabla\varphi\|_H^2(x)\}^{1/2} + C_2 \|\varphi\|_\infty \right)$$

( $\nabla$  denotes the Fréchet derivative). Assume moreover that for some  $p > 1$

$$QW^{2p} \leq C_3^2 W^{2p-2}, \quad 4C_1 C_3 < 1.$$

Let  $\mathbf{Q}^x$  be the law of the Markov chain  $(X_n)_{n \geq 0}$  with transition kernel  $Q$  and  $X_0 = x$ . Then there exists a coupling  $\mathbf{Q}^{x,x'} \in \mathcal{C}(\mathbf{Q}^x, \mathbf{Q}^{x'})$  such that

$$\mathbf{Q}^{x,x'} \left[ \|X_n - X'_n\|_H \leq C_2^{-1} 2^{-(n+1)} \text{ for all } n \geq 1 \right] \geq \frac{1}{2}$$

whenever  $\|x - x'\|_H \leq (4C_2 R)^{-1}$ ,  $W^p(x) \leq R$ ,  $W^p(x') \leq R$  for some  $R > 1$ . Moreover, the map  $(x, x') \mapsto \mathbf{Q}^{x,x'}$  can be chosen to be measurable.

PROOF. We have simply rephrased the proofs of Propositions 5.5 and 4.12 in [17], making explicit choices for the constants involved.  $\square$

Denote by  $P_0(x, \cdot) = \mathbf{P}^x[X_1 \in \cdot]$  the transition kernel of  $(X_n)_{n \geq 0}$ . To verify the assumptions of Theorem 5.8, we require the following deep result. This is the combined statement of Proposition 4.15 and Lemma A.1 in [16].

THEOREM 5.9. *For every  $\eta > 0$  and  $C_1 > 0$ , there exists  $C_2 > 0$  so that*

$$\|\nabla P_0\varphi(x)\|_H \leq \exp(\eta\|x\|_H^2) \left( C_1 \{P_0\|\nabla\varphi\|_H^2(x)\}^{1/2} + C_2 \|\varphi\|_\infty \right)$$

for all  $\varphi \in C^1(H)$  and  $x \in H$ . Moreover, there exist constants  $\eta_0 > 0$  and  $C_3 > 0$  such that for every  $0 < \eta' \leq \eta_0$ ,  $x \in H$ , and  $n \geq 1$  we have

$$\mathbf{E}^x[\exp(\eta' \|X_n\|_H^2)] \leq C_3^2 \exp(\eta' e^{-\nu n \delta} \|x\|_H^2).$$

Finally, we require the following reachability lemma [12, Lemma 3.1].

LEMMA 5.10. *For any  $R_1, R_2 > 0$ , there exist  $n \geq 1$  and  $q > 0$  such that*

$$\inf_{\|x\|_H \leq R_1} \mathbf{P}^x[\|X_n\|_H \leq R_2] \geq q > 0.$$

Using these results, we can now obtain the following asymptotic coupling.

COROLLARY 5.11. *There exists  $\alpha > 0$  such that*

$$\forall x, x' \in H, \quad \exists \mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'}) \quad \text{s.t.} \quad \mathbf{Q} \left[ \sum_{n=1}^{\infty} \|X_n - X'_n\|_{H^1}^2 < \infty \right] \geq \alpha.$$

PROOF. Let  $W(x) = \exp(\eta \|x\|_H^2)$  with  $\eta = (1 - e^{-\nu \delta})\eta_0/2$ , and define  $p = (1 - e^{-\nu \delta})^{-1}$  and  $C_1 = 1/8C_3$  (here  $\eta_0$  and  $C_3$  are as in Theorem 5.9). Defining  $C_2$  as in Theorem 5.9, it is easily verified that the assumptions of Theorem 5.8 are satisfied for  $Q = P_0$ . Therefore, for any  $u, u' \in H$  such that  $\|u\|_H \leq R_2$  and  $\|u'\|_H \leq R_2$ , there exists  $\mathbf{Q}^{u,u'} \in \mathcal{C}(\mathbf{P}^u, \mathbf{P}^{u'})$  such that

$$\mathbf{Q}^{u,u'} \left[ \sup_{n \geq 1} 2^n \|X_n - X'_n\|_H < \infty \right] \geq \frac{1}{2},$$

where we defined the constant  $R_2 = \sqrt{(2 \log 2)/\eta_0} \wedge (16C_2)^{-1}$ . On the other hand, define the constant  $R_1 = \sqrt{1 + (\log 2 + 2 \log C_3)/\eta_0}$ . Then by Lemma 5.10, there exist  $q > 0$  and  $n_2 \geq 1$  (depending on  $R_1$  and  $R_2$  only) such that

$$\inf_{\|x\|_H \leq R_1} \mathbf{P}^x[\|X_{n_2}\|_H \leq R_2] \geq q > 0.$$

From now on, let us fix  $x, x' \in H$ . Define  $n_1 = 2 \log(\|x\|_H \vee \|x'\|_H)/\nu \delta$ . Then  $\mathbf{E}^u[\exp(\eta_0 \|X_{n_1}\|_H^2)] \leq C_3^2 \exp(\eta_0)$  for  $u = x, x'$  by Theorem 5.9. Using Chebyshev's inequality, we obtain  $\mathbf{P}^u[\|X_{n_1}\|_H \leq R_1] \geq 1/2$  for  $u = x, x'$ .

We now construct the coupling  $\mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'})$  such that

$$\begin{aligned} \mathbf{Q}[X_{0,n_1+n_2}, X'_{0,n_1+n_2} \in \cdot] &= \mathbf{P}^x|_{\mathcal{F}_{0,n_1+n_2}} \otimes \mathbf{P}^{x'}|_{\mathcal{F}_{0,n_1+n_2}}, \\ \mathbf{Q}[X_{n_1+n_2,\infty}, X'_{n_1+n_2,\infty} \in \cdot | \mathcal{F}_{0,n_1+n_2}] &= \mathbf{Q}^{X_{n_1+n_2}, X'_{n_1+n_2}}. \end{aligned}$$

Setting  $\alpha = q^2/8$  (which does not depend on  $x, x'$ ), it is now easily seen that

$$\mathbf{Q} \left[ \sup_{n \geq 1} 2^n \|X_n - X'_n\|_H < \infty \right] \geq \alpha > 0.$$

It remains to strengthen the  $\|\cdot\|_H$ -norm to  $\|\cdot\|_{H^1}$  in this expression. To this end, we employ an interpolation argument. Recall the interpolation inequality  $\|u\|_{H^1} \leq \|u\|_H^{1/2} \|u\|_{H^2}^{1/2}$  (for example, [20, Property 1.1.4]). Therefore, in order to complete the proof, it evidently suffices to show that

$$\mathbf{P}^u \left[ \sum_{n=1}^{\infty} 2^{-n} \|X_n\|_{H^2} < \infty \right] = 1 \quad \text{for all } u \in H.$$

But as we assume that only finitely many Fourier modes are forced, we have  $\mathbf{E}^u[\|X_n\|_{H^2}^2] \leq C(1 + \mathbf{E}^u[\|X_{n-1}\|_H^m])$  for some constants  $m \geq 1$  and  $C > 0$  independent of  $n$  by a standard Sobolev estimate [20, Proposition 2.4.12]. As  $\sup_n \mathbf{E}^u[\|X_n\|_H^m] < \infty$  by Theorem 5.9, the result follows readily.  $\square$

We can now verify the assumptions of Theorem 4.13. Note that for any  $u \in H^2$ , we have  $\|u\|_{\infty} \lesssim \|u\|_{H^2}$  by the Sobolev embedding theorem. In particular,  $\|\mathcal{K}v\|_{\infty} \lesssim \|v\|_{H^1}$  for any  $v \in H^1$ . We can therefore easily compute

$$\int \{ \sqrt{g(x, y)} - \sqrt{g(x', y)} \}^2 \varphi(dy) \leq C \|x - x'\|_{H^1}^2 \quad \text{for all } x, x' \in H$$

as in Lemma 5.3. In view of Corollary 5.11, we have verified the assumptions of Theorem 4.13 for  $\tilde{d}(x, y) = \|x - y\|_{H^1}$ . We therefore conclude that

$$\|\pi_n^{\mu} - \pi_n^{\nu}\|_{\text{BL}} \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^{\gamma}\text{-probability} \quad \text{for all } \mu, \nu, \gamma \in \mathcal{P}(E),$$

that is, we have established filter stability in the bounded-Lipschitz norm.

**5.3. Stochastic spin systems.** We now turn to an example of an essentially different nature: we consider a stochastic spin system with counting observations (this could serve a stylized model, for example, of photocount data from optical observations of a chain of ions in a linear trap). In this setting, the unobserved process  $(x_t)_{t \geq 0}$  describes the configuration of spins in one dimension; that is,  $x_t$  takes values in the space  $E = \{0, 1\}^{\mathbb{Z}}$ , where  $x_t^i \in \{0, 1\}$  denotes the state of spin  $i \in \mathbb{Z}$  at time  $t \geq 0$ . The observations  $(y_t)_{t \geq 0}$  are modeled by a counting process, so that  $y_t$  takes values in  $F = \mathbb{Z}_+$ .

To define the dynamics of  $(x_t)_{t \geq 0}$ , we introduce a function  $c_i : E \rightarrow ]0, \infty[$  for every spin  $i \in \mathbb{Z}$ . We interpret  $c_i(\sigma)$  as the rate at which spin  $i$  flips when the system is in the configuration  $\sigma$ . We will make the following assumptions.

ASSUMPTION 5.12. We assume the flip rates are (a) uniformly bounded:  $\sup_{i,\sigma} c_i(\sigma) < \infty$ ; (b) finite range:  $c_i(\sigma)$  depends only on  $\sigma_j$ ,  $|i-j| \leq R < \infty$ ; (c) translation invariant:  $c_i(\sigma) = c_{i+1}(\sigma')$  if  $\sigma_j = \sigma'_{j+1}$  for all  $i, j$ .

The interpretation of  $c_i(\sigma)$  is made precise by defining the pregenerator

$$\mathcal{L}f(\sigma) = \sum_{i \in \mathbb{Z}} c_i(\sigma) \{f(\sigma^i) - f(\sigma)\} \quad \text{for } \sigma \in E, f \in \mathcal{C},$$

where  $\sigma_j^i = \sigma_j$  for  $j \neq i$  and  $\sigma_i^i = 1 - \sigma_i$  and  $\mathcal{C}$  is the space of cylinder functions on  $E$ . Then the closure of  $\mathcal{L}$  in  $C(E)$  is the generator of a Markov semigroup [24, Ch. III], and we let  $(x_t)_{t \geq 0}$  be the associated Markov process. To ensure good ergodic properties of  $(x_t)_{t \geq 0}$ , we will assume the following.

ASSUMPTION 5.13. The spin system  $(x_t)_{t \geq 0}$  is reversible with respect to some probability  $\lambda$ . Moreover, the flip rates are attractive: if  $\sigma \leq \sigma'$ , then we have  $c_i(\sigma) \leq c_i(\sigma')$  if  $\sigma_i = \sigma'_i = 0$  and  $c_i(\sigma) \geq c_i(\sigma')$  if  $\sigma_i = \sigma'_i = 1$ .

It is known that under our assumptions,  $\lambda$  is necessarily a Gibbs measure [24, Theorem IV.2.13] (so this is a stochastic Ising model). The attractive dynamics will tend to make neighboring spins agree; in this setting,  $(x_t)_{t \geq 0}$  admits  $\lambda$  as its unique invariant measure [24, Theorem IV.3.13].

To define the observations, we will fix a strictly positive continuous function  $h : E \rightarrow ]0, \infty[$ . The conditional law of  $(y_t)_{t \geq 0}$  given  $(x_t)_{t \geq 0}$  is modeled as an inhomogeneous Poisson process with rate  $\lambda_t = h(x_t)$ .

LEMMA 5.14. *The pair  $(x_t, y_t)_{t \geq 0}$  defines a nondegenerate Markov additive process in  $\{0, 1\}^{\mathbb{Z}} \times \mathbb{Z}_+$  that admits a unique invariant probability  $\lambda$ .*

PROOF. That  $(x_t, y_t)_{t \geq 0}$  defines a Markov additive process is evident, and the existence of a unique invariant probability under the assumptions of this section was stated above. To establish nondegeneracy it suffices to note that as  $h$  is strictly positive, the conditional law of  $y_{0,\delta}$  given  $x_{0,\delta}$  is equivalent to the law  $\varphi_\delta$  of a unit-rate Poisson process by [25, Theorem 19.4].  $\square$

We will require below the stronger assumption that the observation function  $h$  is Lipschitz continuous with respect to a suitable metric. Note that for any choice of scalars  $\alpha_i > 0$  (for  $i \in \mathbb{Z}$ ) such that  $\sum_i \alpha_i < \infty$ , the quantity

$$d(\sigma, \sigma') = \sum_{i \in \mathbb{Z}} \alpha_i \mathbf{1}_{\sigma_i \neq \sigma'_i}, \quad \sigma, \sigma' \in E = \{0, 1\}^{\mathbb{Z}},$$

metrizes the product topology of  $\{0, 1\}^{\mathbb{Z}}$ . We will assume throughout this section that  $h$  is Lipschitz with respect to  $d$  for a suitable choice of  $(\alpha_i)_{i \in \mathbb{Z}}$ .

We now aim to establish stability of the filter. As we can naturally write  $E = \prod_{i \in I} E^i$  with  $I = \mathbb{Z}$  and  $E^i = \{0, 1\}$ , we are in the setting of Theorem 4.20. To apply it, we must establish the local mixing property. To this end we will use two essential tools: a uniform ergodicity result due to Holley and Stroock [18], and the well-known Wasserstein coupling [24, section III.1].

PROPOSITION 5.15.  *$(x_t, y_t)_{t \geq 0}$  is locally mixing.*

PROOF. Fix a point  $x \in E$  and a finite subset  $J \subseteq \mathbb{Z}$ ,  $|J| < \infty$  throughout the proof, and let  $\mathcal{F}_{s,t}^J = \sigma\{x_{s,t}^J, y_{s,t}\}$ . It evidently suffices to show that

$$\|\mathbf{P}^x - \mathbf{P}^o\|_{\mathcal{F}_{t,\infty}^J} \xrightarrow{t \rightarrow \infty} 0,$$

where  $o \in E$  is the zero configuration. Let  $\mathbf{Q} \in \mathcal{C}(\mathbf{P}^o, \mathbf{P}^x)$  be the Wasserstein coupling [24, section III.1]. As obviously  $o \leq x$ , we have [24, Theorem III.1.5]

$$x_t \leq x'_t \quad \text{for all } t \geq 0 \quad \mathbf{Q}\text{-a.s.}$$

To proceed, we recall a result of Holley and Stroock [18, Theorem 0.4]:

$$\sup_{\sigma, \sigma' \in E} |\mathbf{P}^\sigma[x_t^0 = 1] - \mathbf{P}^{\sigma'}[x_t^0 = 1]| \leq C e^{-\gamma t} \quad \text{for all } t \geq 0$$

for some constants  $C, \gamma > 0$ . By translation invariance, it follows that

$$\sup_{i \in \mathbb{Z}} \sup_{\sigma, \sigma' \in E} |\mathbf{P}^\sigma[x_t^i = 1] - \mathbf{P}^{\sigma'}[x_t^i = 1]| \leq C e^{-\gamma t} \quad \text{for all } t \geq 0.$$

Therefore, by monotonicity,

$$\mathbf{Q}[x_t^i \neq x_t^{i'}] = \mathbf{E}_{\mathbf{Q}}[x_t^{i'} - x_t^i] \leq C e^{-\gamma t} \quad \text{for all } t \geq 0, i \in \mathbb{Z}.$$

Using the Lipschitz property of  $h$ , it follows easily that

$$\mathbf{E}_{\mathbf{Q}} \left[ \int_0^\infty |h(x_t) - h(x'_t)| dt \right] < \infty \quad \text{and} \quad \mathbf{E}_{\mathbf{Q}} \left[ \int_0^\infty \mathbf{1}_{x_t^J \neq x_t^{J'}} dt \right] < \infty.$$

We claim that the second inequality implies that  $x_t^J = x_t^{J'}$  for all  $t$  sufficiently large  $\mathbf{Q}$ -a.s.; we postpone the verification of this claim until the end of the proof. Assuming the claim, we now complete the proof of local mixing.



Let us extend the Wasserstein coupling  $\mathbf{Q}$  to the triple  $(x_t, x'_t, y_t)_{t \geq 0}$  by letting  $(y_t)_{t \geq 0}$  be an inhomogeneous Poisson process with rate  $\lambda_t = h(x_t)$  conditionally on  $(x_t, x'_t)_{t \geq 0}$ . Define for any  $t \geq 0$  the random variable

$$\Lambda_t = \exp \left[ \int_t^\infty \{ \log h(x'_{s-}) - \log h(x_{s-}) \} dy_s - \int_t^\infty \{ h(x'_s) - h(x_s) \} ds \right].$$

Applying [25, Lemma 19.6] conditionally yields  $\mathbf{E}[\Lambda_t] = 1$  for all  $t \geq 0$ . By the change of measure theorem for Poisson processes [25, Theorem 19.4]

$$\mathbf{P}^o(A) = \mathbf{E}_{\mathbf{Q}}[\mathbf{1}_A(x_{t,\infty}^J, y_{t,\infty})], \quad \mathbf{P}^x(A) = \mathbf{E}_{\mathbf{Q}}[\mathbf{1}_A(x_{t,\infty}^{J'}, y_{t,\infty}) \Lambda_t]$$

for any  $A \in \mathcal{F}_{t,\infty}^J$ . Thus we can estimate

$$\|\mathbf{P}^x - \mathbf{P}^o\|_{\mathcal{F}_{t,\infty}^J} \leq \mathbf{Q}[x_{t,\infty}^J \neq x_{t,\infty}^{J'}] + \mathbf{E}_{\mathbf{Q}}[|1 - \Lambda_t|] \xrightarrow{t \rightarrow \infty} 0,$$

where the convergence follows by Scheffé's lemma and the above claim.

It remains to prove the claim. To this end, define the stopping times  $\tau_0 = 0$  and  $\tau'_n = \inf\{t \geq \tau_n : x_t^J = x_t^{J'}\}$  and  $\tau_{n+1} = \inf\{t \geq \tau'_n : x_t^J \neq x_t^{J'}\}$  for  $n \geq 0$ . By right-continuity  $\tau'_n > \tau_n$  on  $\{\tau_n < \infty\}$  for all  $n \geq 1$ , and

$$\mathbf{E}_{\mathbf{Q}} \left[ \sum_{n=0}^{\infty} (\tau'_n - \tau_n) \mathbf{1}_{\tau_n < \infty} \right] = \mathbf{E}_{\mathbf{Q}} \left[ \int_0^\infty \mathbf{1}_{x_t^J \neq x_t^{J'}} dt \right] < \infty.$$

Now denote by  $\tau''_n = \inf\{t \geq \tau_n : (x_t^J, x_t^{J'}) \neq (x_{\tau_n}^J, x_{\tau_n}^{J'})\}$ . As the Wasserstein coupling is itself a particle system with uniformly bounded rates, it is a routine exercise to verify that there exists a constant  $c > 0$  such that

$$\mathbf{E}_{\mathbf{Q}}[\tau'_n - \tau_n | \mathcal{F}_{\tau_n}] \mathbf{1}_{\tau_n < \infty} \geq \mathbf{E}_{\mathbf{Q}}[\tau''_n - \tau_n | \mathcal{F}_{\tau_n}] \mathbf{1}_{\tau_n < \infty} \geq c \mathbf{1}_{\tau_n < \infty} \quad \mathbf{Q}\text{-a.s.}$$

It follows that  $\tau_n = \infty$  eventually  $\mathbf{Q}$ -a.s., which yields the claim.  $\square$

We have now verified all the assumptions of Theorem 4.20. Thus we have

$$\|\pi_t^\mu - \pi_t^\nu\|_{\mathcal{E}^J} \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^\gamma\text{-probability for all } |J| < \infty, \mu, \nu, \gamma \in \mathcal{P}(E),$$

where  $\mathcal{E}^J \subseteq \mathcal{B}(E)$  be the cylinder  $\sigma$ -field generated by the spins  $J$ .

**REMARK 5.16.** The proof just given works only in one spatial dimension  $I = \mathbb{Z}$ . In a higher-dimensional lattice  $I = \mathbb{Z}^d$ , the (unconditional) ergodic theory of the spin system becomes much more subtle as phase transitions typically appear. A Dobrushin-type sufficient condition for local mixing in any dimension is given by Föllmer [15] for a related discrete-time model. With some more work, this approach can also be applied to continuous time spin systems in the high-temperature regime (for example, by showing that the requisite bounds hold for spatially truncated and time-discretized models, uniformly in the truncation and discretization parameters).

5.4. *Stochastic differential delay equations.* Our final example is concerned with filtering in stochastic differential delay equations. Time delays arise naturally in various engineering and biological applications, and the corresponding filtering problem has been investigated by a number of authors [44, 23, 4]. In particular, some results on filter stability for linear delay equations have been investigated in [44, 23] by means of the associated Kalman equations. We tackle here the much more difficult nonlinear case.

Fix throughout this section a delay  $r \in \mathbb{R}_+$ . Following [4], for example, we introduce the following nonlinear filtering model with time delay. The unobserved process is defined by the stochastic differential delay equation

$$dx(t) = f(x_t) dt + g(x_t) dW_t,$$

where  $(W_t)_{t \geq 0}$  is  $m$ -dimensional Brownian motion,  $x(t)$  takes values in  $\mathbb{R}^n$ , and we have introduced the notation  $x_t = (x(t+s))_{s \in [-r, 0]} \in C([-r, 0]; \mathbb{R}^n)$ . The  $\mathbb{R}^d$ -valued observations are defined by the white noise model

$$dy_t = h(x_t) dt + dB_t,$$

where  $(B_t)_{t \geq 0}$  is  $d$ -dimensional Brownian motion independent of  $(x_t)_{t \geq 0}$ .

In the following, we will exploit heavily the ergodicity results for stochastic delay equations established in [17]. To this end, we work under the following assumptions. Here and in the sequel, we denote by  $\|x\| = \sup_{t \in [-r, 0]} |x(t)|$  for  $x \in C([-r, 0]; \mathbb{R}^n)$ , and by  $|M|^2 = \text{Tr}[MM^*]$  for any matrix  $M$ .

ASSUMPTION 5.17. Assume (a) there exists  $g^{-1} : C([-r, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^{m \times n}$  with  $\|g^{-1}\|_\infty < \infty$  and  $g(x)g^{-1}(x) = \text{Id}_m$  for all  $x$ ; (b)  $f$  is continuous and bounded on bounded subsets of  $C([-r, 0]; \mathbb{R}^n)$ ; (c) for all  $x, y$ , we have  $2\langle f(x) - f(y), x(0) - y(0) \rangle^+ + |g(x) - g(y)|^2 + |h(x) - h(y)|^2 \leq L\|x - y\|^2$ .

Under this assumption, the equation for  $(x(t))_{t \geq 0}$  possesses a unique strong solution for any initial condition  $(x(t))_{t \in [-r, 0]}$  such that  $(x_t)_{t \geq 0}$  is a  $C([-r, 0]; \mathbb{R}^n)$ -valued strong Markov process [17]. Thus the pair  $(x_t, y_t)$  is evidently a nondegenerate Markov additive process in  $C([-r, 0]; \mathbb{R}^n) \times \mathbb{R}^d$ .

The previous assumption does not ensure the existence of an invariant probability. Rather than imposing explicit sufficient conditions (see, for example, [14, 9]), it will suffice simply to assume that such a probability exists.

ASSUMPTION 5.18.  $(x_t)_{t \geq 0}$  admits an invariant probability  $\lambda$ .

To establish stability of the filter, we will apply Theorem 4.23. To construct an asymptotic coupling, the key result that we will use is the following.

**THEOREM 5.19.** *For every  $x, x' \in C([-r, 0]; \mathbb{R}^n)$ , there exists a coupling  $\mathbf{Q}^{x, x'} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'})$  such that the map  $(x, x') \mapsto \mathbf{Q}^{x, x'}$  is measurable and*

$$\inf_{\|x\|, \|x'\| \leq R} \mathbf{Q}^{x, x'} \left[ \sup_{t \geq 0} e^t \|x_t - x'_t\| < \infty \right] = \beta_R > 0 \quad \text{for every } R < \infty.$$

We postpone the proof of this result to the end of this section. Let us now show how the result can be used to verify the assumptions of Theorem 4.23.

We first construct the asymptotic coupling. Let us choose  $R > 0$  such that  $\lambda(\|x\| < R) > 1/2$ . By [17, Theorem 3.7] and the Portmanteau theorem, we have  $\mathbf{P}^x[\|x_t\| < R] \geq 1/2$  eventually as  $t \rightarrow \infty$  for every  $x \in C([-r, 0]; \mathbb{R}^n)$ . Let  $\alpha = \beta_R/4$ . Given any  $x, x' \in C([-r, 0]; \mathbb{R}^n)$ , we now construct a coupling  $\mathbf{Q} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'})$  as follows. First, choose  $s > 0$  such that  $\mathbf{P}^x[\|x_s\| < R] \geq 1/2$  and  $\mathbf{P}^{x'}[\|x_s\| < R] \geq 1/2$ . We then define the coupling  $\mathbf{Q}$  such that

$$\mathbf{Q}[x_{0,s}, x'_{0,s} \in \cdot] = \mathbf{P}^x|_{\mathcal{F}_{0,s}} \otimes \mathbf{P}^{x'}|_{\mathcal{F}_{0,s}}, \quad \mathbf{Q}[x_{s,\infty}, x'_{s,\infty} \in \cdot | \mathcal{F}_{0,s}] = \mathbf{Q}^{x_s, x'_s}.$$

By construction, we have

$$\mathbf{Q} \left[ \sup_{t \geq 0} e^t \|x_t - x'_t\| < \infty \right] \geq \alpha.$$

Thus we have evidently verified the first assumption of Theorem 4.23 for  $\tilde{d}(x, x') = \|x - x'\|$  and  $\Delta = 1$  (for example). On the other hand, the second assumption follows easily as in Lemma 5.3, as we have assumed the Lipschitz property of  $h$ . Thus we have verified the assumptions of Theorem 4.23, so

$$\|\pi_t^\mu - \pi_t^\nu\|_{\text{BL}} \xrightarrow{t \rightarrow \infty} 0 \quad \text{in } \mathbf{P}^\gamma\text{-probability} \quad \text{for all } \mu, \nu, \gamma \in \mathcal{P}(C([-r, 0]; \mathbb{R}^n)),$$

that is, we have established filter stability in the bounded-Lipschitz norm.

It remains to prove Theorem 5.19. This is a direct extension of the proof of Theorem 3.1 in [17]; we finish the section by sketching the necessary steps.

**PROOF OF THEOREM 5.19.** In the proof of [17, Theorem 3.1] a kernel  $(x, x') \mapsto \tilde{\mathbf{Q}}^{x, x'}$  was constructed on  $\Omega \times \Omega$  with the following properties. First, there exists a constant  $\gamma > 0$  independent of  $x, x'$  such that

$$\tilde{\mathbf{Q}}^{x, x'} \left[ \sup_{t \geq 0} e^t \|x_t - x'_t\| < \infty \right] \geq \gamma \quad \text{for all } x, x' \in C([-r, 0]; \mathbb{R}^n).$$

Second, there is a  $\tilde{\mathbf{Q}}^{x, x'}$ -Brownian motion  $(\tilde{W}_t)_{t \geq 0}$  and an adapted process  $(z_t)_{t \geq 0}$  that satisfies  $\int_0^\infty |z_t|^2 dt \leq C\|x - x'\|^2$   $\tilde{\mathbf{Q}}^{x, x'}$ -a.s. such that

$$\tilde{\mathbf{Q}}^{x, x'}[x_{0,\infty} \in A] = \mathbf{P}^x(A), \quad \tilde{\mathbf{Q}}^{x, x'}[\mathbf{1}_A(x'_{0,\infty})\Lambda] = \mathbf{P}^{x'}(A)$$

for every measurable set  $A$ , where  $\Lambda$  is the Girsanov density

$$\Lambda = \exp \left[ \int_0^\infty z_t d\tilde{W}_t - \frac{1}{2} \int_0^\infty |z_t|^2 dt \right].$$

Let  $\mathbf{R}^{x,x'} \in \mathcal{C}(\tilde{\mathbf{Q}}^{x,x'}, \mathbf{P}^{x'})$  be the coupling maximizing  $\mathbf{R}^{x,x'}[x'_{0,\infty} = x''_{0,\infty}]$ . It is classical that  $2\mathbf{R}^{x,x'}[x'_{0,\infty} \neq x''_{0,\infty}] = \|\tilde{\mathbf{Q}}^{x,x'}[x'_{0,\infty} \in \cdot] - \mathbf{P}^{x'}\|$  and that the maximal coupling can be chosen to be measurable in  $x, x'$  (by the existence of a measurable version of the Radon-Nikodym density between kernels [10, V.58]). As  $\int_0^\infty |z_t|^2 dt \leq C\|x - x'\|^2$   $\tilde{\mathbf{Q}}^{x,x'}$ -a.s., we can choose  $\delta > 0$  sufficiently small that  $\mathbf{R}^{x,x'}[x'_{0,\infty} \neq x''_{0,\infty}] \leq \gamma/2$  whenever  $\|x - x'\| \leq \delta$ . Then evidently

$$\mathbf{R}^{x,x'} \left[ \sup_{t \geq 0} e^t \|x_t - x''_t\| < \infty \right] \geq \frac{\gamma}{2} \quad \text{whenever } \|x - x'\| \leq \delta.$$

Now define for any  $x, x' \in C([-r, 0]; \mathbb{R}^n)$  the measure  $\mathbf{Q}^{x,x'}$  such that

$$\begin{aligned} \mathbf{Q}^{x,x'}[x_{0,2r}, x'_{0,2r} \in \cdot] &= \mathbf{P}^x|_{\mathcal{F}_{0,2r}} \otimes \mathbf{P}^{x'}|_{\mathcal{F}_{0,2r}}, \\ \mathbf{Q}^{x,x'}[x_{2r,\infty}, x'_{2r,\infty} \in \cdot | \mathcal{F}_{0,2r}] &= \mathbf{R}^{x_{2r}, x'_{2r}}[x_{0,\infty}, x'_{0,\infty} \in \cdot]. \end{aligned}$$

Then  $\mathbf{Q}^{x,x'} \in \mathcal{C}(\mathbf{P}^x, \mathbf{P}^{x'})$ ,  $(x, x') \mapsto \mathbf{Q}^{x,x'}$  is measurable, and

$$\inf_{\|x\|, \|x'\| \leq R} \mathbf{Q}^{x,x'} \left[ \sup_{t \geq 0} e^t \|x_t - x'_t\| < \infty \right] \geq \frac{\gamma}{2} \left( \inf_{\|x\| \leq R} \mathbf{P}^x \left[ \|x_{2r}\| \leq \frac{\delta}{2} \right] \right)^2.$$

It remains to note that the right-hand side is positive by [17, Lemma 3.8].  $\square$

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